Measurement and Research Department Reports 2003-2

Equivalent Mirid Models

Gunter Maris
Timo Bechger

Citogroep

Measurement and Research Department Reports 2003-2

EQUIVALENT MIRID MODELS

Gunter Maris Timo Bechger

CITO, NATIONAL INSTITUTE FOR EDUCATIONAL MEASUREMENT

ARNHEM

Citogroep
Arnhem, March 3, 2003

Abstract

In this paper we show that different componential theories about an item set may lead to equivalent MIRID models. Furthermore, we provide conditions for the identifiability of the MIRID model parameters and it will be shown how the MIRID model relates to the LLTM. These results extend the work of Bechger, Verhelst, and Verstralen (2001), and Bechger, Verstralen, and Verhelst (2002).

Key words: MIRID, LLTM, Identifiability

MIRID is an abbreviation for "model with internal restrictions on the item difficulties". The MIRID model is proposed by Butter (1994), Butter, De Boeck, and Verhelst (1998) as a model for sets of items consisting of some items that require single operations and one or more items which require a number of these operations simultaneously. Suppose, for example, that subjects have been administered three math items. The first item requires an addition, the second item requires a subtraction, and the third item combines the addition and subtraction operations of the two other ones.

The items that refer to a single mental operation (or component) are called *subtasks*, and the item involving a combination of the components is called the *composite task*. It is assumed that for a set of items, the Rasch model is valid with the restriction that the difficulty of the composite items is a linear combination of the difficulties of the subtasks. That is, the MIRID model assumes that the difficulties of the composite tasks consist of a regression on the subtask difficulties of items within the same item family.

If the regression weights were known the model would be a linear logistic test model (LLTM) (Schleiblechner, 1972; Fischer, 1995). In the present situation, however, they are considered as model parameters.

In the present paper we elaborate on Bechger et al. (2002) who note that different theories may give rise to equivalent LLTMs and on Bechger et al. (2001) who deal with parameter identifiability for the larger class of nonlinear logistic test models (NLTM). After giving a more formal definition of the MIRID model (Section 1), we deal with the problem of parameter identifiability in Section 2. In this section we show that for the particular class of NLTMs being considered here, the cumbersome identifiability results from Bechger et al. (2001) can be simplified. In the following section we demonstrate that different componential theories can give rise to equivalent MIRID models. This means, in particular, that it is not clear which items are to be regarded as subtasks and which as composite tasks. This is a serious problem because from a substantive point of view it is much more interesting to test different componential hypotheses against each other rather than against the Rasch model. In Section 4 we consider the relation between the MIRID model and the LLTM. The fifth section provides examples to illustrate the theoretical results and in the sixth section the implications of the results in the paper are discussed.

1. The MIRID Model

Let Y_{pi} be a binary random variable whose realizations indicate the response of the p-th subject ($p = 1, \ldots, N$) to the i-th item ($i = 1, \ldots, n$):

$$Y_{pi} = \begin{cases} 1 & \text{if the response is correct} \\ 0 & \text{if the response is incorrect} \end{cases}$$

A MIRID model can be defined in two steps. First, the binary responses Y_{pi} are assumed to satisfy the Rasch model:

$$P(Y_{pi} = 1 | \theta_p, \beta_i) = \frac{\exp(\theta_p - \beta_i)}{1 + \exp(\theta_p - \beta_i)}$$

where θ_p denotes the ability of the p-th subject and β_i denotes the difficulty of the i-th item. Second, the item difficulties are assumed to satisfy the following linear restriction[1]:

$$\boldsymbol{\beta} = \mathbf{Q}(\boldsymbol{\sigma})_{nm} \boldsymbol{\eta}$$

For instance, with six items consisting of four subtasks and two composite tasks

[1]When possible we suppress in our notation that $\mathbf{Q}$ depends on $\boldsymbol{\sigma}$.

$\mathbf{Q}$ could be the following design matrix:

$$\mathbf{Q} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \sigma_1 & \sigma_2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \sigma_1 & \sigma_2 & 1 \end{bmatrix} \tag{1}$$

In this example, σ_1 and σ_2 are the regression weights, $\eta_1, \ldots, \eta_4$ are the subtask difficulties and η_5 is the intercept of the regression. This MIRID model consists of two so-called item families, each consisting of two subtasks and one composite task.

We see that any MIRID model involves a restriction on the parameter space of the Rasch model. From the fact that the Rasch model is an exponential family model, it follows that the MIRID model is a curved exponential family model. Moreover the sufficient statistic for the ability parameter θ_p of the Rasch model (i.e., number of items correct) remains sufficient in the MIRID model. This implies, among other things, that the conditional maximum likelihood method may be used to estimate $\boldsymbol{\eta}$ and $\boldsymbol{\sigma}$.

Since in the MIRID model the item difficulties of the composite tasks are a linear combination of the difficulties of the subtasks, we can express any MIRID model design matrix $\mathbf{Q}$ as follows by means of row and column permutations:

$$\mathbf{P}_1\mathbf{Q}\mathbf{P}_2 = \begin{bmatrix} \mathbf{I}_{kk} & \mathbf{0}_{k(m-k)} \\ \mathbf{A}(\boldsymbol{\sigma})_{(n-k)k} & \mathbf{B}_{(n-k)(m-k)} \end{bmatrix} \tag{2}$$

where k is the number of subtasks, n is the number of items, m is the number of columns of $\mathbf{Q}$, and $\mathbf{P}_1$ and $\mathbf{P}_2$ are row and column permutation matrices. The matrix $\mathbf{A}(\boldsymbol{\sigma})$ contains the regression structure of the model and the matrix $\mathbf{B}$ contains the intercept structure. In Section 5 we show how the design matrix in Equation 1 can be written in this form.

2. Identifiability of the MIRID model parameters

We now consider whether the MIRID model parameters are identifiable by $\mathbf{Y}$. That is, whether different parameter values correspond to different distributions of $\mathbf{Y}$. We first consider whether $\boldsymbol{\eta}$ and $\boldsymbol{\sigma}$ are identifiable from $\boldsymbol{\beta}$. After that we consider the complication that $\boldsymbol{\beta}$ is not identifiable from $\mathbf{Y}$.

It is easily seen that the parameters corresponding to the subtasks $(\eta_1, \ldots, \eta_k)$ are identifiable from $\boldsymbol{\beta}$. In fact, the basic parameter η_j of the j-th subtask is equal to the difficulty parameter of the corresponding item. We now consider the conditions under which the remaining parameters, the intercept parameters $\eta_{k+1}, \ldots, \eta_m$ and the regression weights $\boldsymbol{\sigma}$, are identifiable from $\boldsymbol{\beta}$. We partition $\boldsymbol{\beta}$ as follows $\boldsymbol{\beta}^T = \begin{bmatrix}\boldsymbol{\beta}_1 & \boldsymbol{\beta}_2\end{bmatrix}^T$, where $\boldsymbol{\beta}_1$ refers to the item difficulties of the subtasks and $\boldsymbol{\beta}_2$ to the item difficulties of the composite tasks. Notice that the MIRID model is a bilinear model. That is, it is linear in every parameter. Hence we can always rewrite $\begin{bmatrix}\mathbf{A}(\boldsymbol{\sigma}) & \mathbf{B}\end{bmatrix}\boldsymbol{\eta} = \boldsymbol{\beta}_2$ as follows:

$$\begin{bmatrix}\mathbf{A}^*(\boldsymbol{\beta}_1) & \mathbf{B}\end{bmatrix} \begin{bmatrix}\boldsymbol{\sigma} \\ \eta_{k+1} \\ \eta_{k+2} \\ \vdots \\ \eta_m\end{bmatrix} = \boldsymbol{\beta}_2 \tag{3}$$

as illustrated in Section 5.1. The expression in Equation 3 has the form of the LLTM since the matrix on the left hand side does not depend on parameters. Hence we can use the result from Bechger et al. (2002, proposition 1) who show that the LLTM model parameters are identifiable iff the design matrix has full column rank. That is, the MIRID model parameters are identifiable at points $\boldsymbol{\beta}$ for which the matrix

$\begin{bmatrix}\mathbf{A}^*(\boldsymbol{\beta}_1) & \mathbf{B}\end{bmatrix}$ has full column rank. Notice that this implies that $\mathbf{B}$ has full column rank and it will be shown in the section hereafter that this implies full column rank of $\mathbf{Q}$.

Having dealt with the identifiability of $\boldsymbol{\eta}$ and $\boldsymbol{\sigma}$ from $\boldsymbol{\beta}$, we now consider the complication that $\boldsymbol{\beta}$ is not identifiable from $\mathbf{Y}$. In the Rasch model, the distribution of $\mathbf{Y}$ remains the same if a constant is added to the ability parameters θ_p and the same constant is subtracted from the item parameters β_i. For the MIRID model this implies that if $\mathbf{Q}(\boldsymbol{\sigma}_1)\boldsymbol{\eta}_1 = \boldsymbol{\beta}$ and $\mathbf{Q}(\boldsymbol{\sigma}_2)\boldsymbol{\eta}_2 = \boldsymbol{\beta} + \mathbf{c}$, the model parameters are not identifiable. For the Rasch model it is common practice to solve this problem by imposing a linear restriction on the item parameters. For instance, the sum of the item difficulties equals zero. In Bechger et al. (2002, Section 2) it is shown that for the LLTM such a linear restriction can be imposed by pre-multiplying the design matrix $\mathbf{Q}$ with a normalization matrix $\mathbf{L}$. The same method applies to the MIRID model.

3. Equivalent MIRID Models

The problem addressed in this section is that *different* componential theories for a set of items may give rise to *equivalent* MIRID models. Two different theories, represented by $\mathbf{Q}_0$ and $\mathbf{Q}_1$, are indistinguishable if given $\boldsymbol{\eta}_0$ and $\boldsymbol{\sigma}_0$ we can always find $\boldsymbol{\eta}_1$ and $\boldsymbol{\sigma}_1$ such that: $\mathbf{Q}_0\boldsymbol{\eta}_0 = \mathbf{Q}_1\boldsymbol{\eta}_1$, and vice versa. In plain words this means that $\mathbf{Q}_0$ and $\mathbf{Q}_1$ span the same column space. If two MIRID models satisfy this condition they are said to be equivalent. It is readily seen that equivalent MIRID models correspond to different ways to parameterize the same linear restriction on the parameter space of the Rasch model. The existence of such equivalent parameterizations implies that we cannot substantively interpret any such parameterization. The main results from this Section, Theorem 1 and 2, elaborate and generalize Theorem 2 from Bechger et al. (2002) which gives similar results for the equivalence of different LLTMs.

Throughout this Section we will assume that both $\mathbf{Q}_0$ and $\mathbf{Q}_1$ have full column rank. It is readily found that $\mathbf{Q}$ has full column rank iff[2] $\mathbf{Q}^{g_1}\mathbf{Q} = \mathbf{I}$. A generalized inverse of $\mathbf{Q}$ has the following form:

$$(\mathbf{P}_1\mathbf{Q}\mathbf{P}_2)^{g_1} = \mathbf{P}_2^{-1}\mathbf{Q}^{g_1}\mathbf{P}_1^{-1} = \begin{bmatrix} \mathbf{I}_{kk} & \mathbf{0}_{k(n-k)} \\ -\mathbf{B}^{g_1}_{(m-k)(n-k)}\mathbf{A}(\boldsymbol{\sigma})_{(n-k)k} & \mathbf{B}^{g_1}_{(m-k)(n-k)} \end{bmatrix} \tag{4}$$

In general the matrix $\mathbf{0}_{k(n-k)}$ can be replaced by any matrix $\mathbf{K}$ that is such that $\mathbf{K}\mathbf{A}(\boldsymbol{\sigma}) = \mathbf{0}$ and $\mathbf{K}\mathbf{B} = \mathbf{0}$. However, we will see below that no generality is lost by assuming $\mathbf{K} = \mathbf{0}$. Up to row and column permutations, $\mathbf{Q}^{g_1}\mathbf{Q}$ can be expressed as follows:

$$(\mathbf{P}_1\mathbf{Q}\mathbf{P}_2)^{g_1}\mathbf{P}_1\mathbf{Q}\mathbf{P}_2 = \mathbf{P}_2^{-1}\mathbf{Q}^{g_1}\mathbf{Q}\mathbf{P}_2 = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}^{g_1}\mathbf{B} \end{bmatrix} .$$

for any admissable $\mathbf{K}$. We see that $\mathbf{Q}$ has full column rank if and only if $\mathbf{B}$ has full column rank. Full column rank of $\mathbf{B}$ and hence of $\mathbf{Q}$ is a necessary condition for the identifiability of the MIRID model parameters. Notice that if $\mathbf{Q}$ has full column rank, $\mathbf{Q}^{g_1} = \left(\mathbf{Q}^T\mathbf{Q}\right)^{-1}\mathbf{Q}^T$ is a particularly simple generalized inverse.

In Theorem 1 we elaborate on the conditions under which MIRID models are equivalent.

Theorem 1. $\mathbf{Q}_0$ and $\mathbf{Q}_1$ span the same column space iff both $\mathbf{Q}_1\mathbf{Q}_1^{g_1}\mathbf{Q}_2 = \mathbf{Q}_2$ and $\mathbf{Q}_2\mathbf{Q}_2^{g_1}\mathbf{Q}_1 = \mathbf{Q}_1$

Proof. Theorem 1.5 in Pringle and Rayner (1971). Note that if $\mathbf{Q}_1\mathbf{Q}_1^{g_1}\mathbf{Q}_2 = \mathbf{Q}_2$ holds for any particular generalized inverse, it holds for all generalized inverses,

[2] $\mathbf{Q}^{g_1}$ denotes a (one-condition) generalized inverse of $\mathbf{Q}$: $\mathbf{Q}\mathbf{Q}^{g_1}\mathbf{Q} = \mathbf{Q}$.

denoted by $\mathbf{X}$:

$$
\begin{aligned}
(\mathbf{Q}_1\mathbf{X})\,\mathbf{Q}_2 &= (\mathbf{Q}_1\mathbf{X})\,(\mathbf{Q}_1\mathbf{Q}_1^{g_1}\mathbf{Q}_2)\\
&= \mathbf{Q}_1\mathbf{Q}_1^{g_1}\mathbf{Q}_2\\
&= \mathbf{Q}_2 \quad ,
\end{aligned}
$$

and hence we may assume $\mathbf{K} = \mathbf{0}$.

Since g_1-inverses of both $\mathbf{Q}_0$ and $\mathbf{Q}_1$ have the simple form in Equation 4, Theorem 1 furnishes an easy method to evaluate whether two MIRID models are equivalent or not. Moreover, it is easily found that, up to row and column permutations, $\mathbf{Q}\mathbf{Q}^{g_1}$ has the following form:

$$
\mathbf{P}_1^{-1}\mathbf{Q}\mathbf{Q}^{g_1}\mathbf{P}_1 = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ (\mathbf{I}-\mathbf{B}\mathbf{B}^{g_1})\mathbf{A}(\boldsymbol{\sigma}) & \mathbf{B}\mathbf{B}^{g_1} \end{bmatrix}
$$

However, the theorem is of little help if we want to construct equivalent MIRID models. This problem is addressed in the following theorem:

Theorem 2. $\mathbf{Q}_1 = \mathbf{Q}_2\mathbf{P}$, where $\mathbf{P}$ is non-singular, iff both $\mathbf{Q}_1\mathbf{Q}_1^{g_1}\mathbf{Q}_2 = \mathbf{Q}_2$ and $\mathbf{Q}_2\mathbf{Q}_2^{g_1}\mathbf{Q}_1 = \mathbf{Q}_1$

Proof. (IF) Choose $\mathbf{P} = \mathbf{Q}_2^{g_1}\mathbf{Q}_1$. Hence, we only have to show that $\mathbf{P}$ is non-singular. Since both $\mathbf{Q}_1$ and $\mathbf{Q}_2$ have full column rank we easily obtain the following, with $\mathbf{P}^{-1} = \mathbf{Q}_1^{g_1}\mathbf{Q}_2$:

$$
\mathbf{P}\mathbf{P}^{-1} = \mathbf{Q}_2^{g_1}\mathbf{Q}_1\mathbf{Q}_1^{g_1}\mathbf{Q}_2 = \mathbf{Q}_2^{g_1}\mathbf{Q}_2 = \mathbf{I}
$$

and

$$
\mathbf{P}^{-1}\mathbf{P} = \mathbf{Q}_1^{g_1}\mathbf{Q}_2\mathbf{Q}_2^{g_1}\mathbf{Q}_1 = \mathbf{Q}_1^{g_1}\mathbf{Q}_1 = \mathbf{I}
$$

(ONLY IF) If $\mathbf{Q}_1 = \mathbf{Q}_2\mathbf{P}$ we obtain the following by pre-multiplying with $\mathbf{Q}_2\mathbf{Q}_2^{g_1}$:

$$\mathbf{Q}_2\mathbf{Q}_2^{g_1}\mathbf{Q}_1 = \mathbf{Q}_2\mathbf{Q}_2^{g_1}\mathbf{Q}_2\mathbf{P} = \mathbf{Q}_2\mathbf{P} = \mathbf{Q}_1$$

In the same way we obtain that $\mathbf{Q}_1\mathbf{Q}_1^{g_1}\mathbf{Q}_2 = \mathbf{Q}_2$ using the condition that $\mathbf{P}$ is non-singular.

Hence, we can construct equivalent MIRID models by choosing a non-singular matrix $\mathbf{P}$ as will be illustrated in Section 5.2.

It two MIRID models are equivalent, the relation between the parameters $\boldsymbol{\eta}_1$ and $\boldsymbol{\eta}_2$ is easily obtained. If $\mathbf{Q}_1 = \mathbf{Q}_2\mathbf{P}$, then $\mathbf{Q}_1\boldsymbol{\eta}_1 = \mathbf{Q}_2\mathbf{P}\boldsymbol{\eta}_1 = \mathbf{Q}_2\boldsymbol{\eta}_2$. Hence $\boldsymbol{\eta}_2 = \mathbf{P}\boldsymbol{\eta}_1$ and since $\mathbf{P}$ is non-singular $\boldsymbol{\eta}_1 = \mathbf{P}^{-1}\boldsymbol{\eta}_2$.

Using the results of this section we can approach the problem of parameter identifiability from a different angle. In particular, the parameters are identifiable iff:

$$\mathbf{Q}(\boldsymbol{\sigma}_1)\boldsymbol{\eta}_1 = \mathbf{Q}(\boldsymbol{\sigma}_2)\boldsymbol{\eta}_2$$

implies that $\boldsymbol{\eta}_1 = \boldsymbol{\eta}_2$ and $\boldsymbol{\sigma}_1 = \boldsymbol{\sigma}_2$. If the parameters are not identifiable we say that the model is self-equivalent in the sense that $\mathbf{Q}(\boldsymbol{\sigma}_1)\boldsymbol{\eta}_1 = \mathbf{Q}(\boldsymbol{\sigma}_2)\boldsymbol{\eta}_2$ holds for $(\boldsymbol{\eta}_1, \boldsymbol{\sigma}_1) \neq (\boldsymbol{\eta}_2, \boldsymbol{\sigma}_2)$. To evaluate whether a model is self-equivalent at distinct parameter points we need to check the conditions of Theorem 1. We find that the condition $\mathbf{Q}(\boldsymbol{\sigma}_1)\mathbf{Q}(\boldsymbol{\sigma}_1)^{g_1}\mathbf{Q}(\boldsymbol{\sigma}_2) = \mathbf{Q}(\boldsymbol{\sigma}_2)$ can be written as follows:

$$\begin{aligned}
\mathbf{Q}(\boldsymbol{\sigma}_1)\mathbf{Q}(\boldsymbol{\sigma}_1)^{g_1}\mathbf{Q}(\boldsymbol{\sigma}_2) &= \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ (\mathbf{I} - \mathbf{B}\mathbf{B}^{g_1})\mathbf{A}(\boldsymbol{\sigma}_1) & \mathbf{B}\mathbf{B}^{g_1} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{A}(\boldsymbol{\sigma}_2) & \mathbf{B} \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{A}(\boldsymbol{\sigma}_1) - \mathbf{B}\mathbf{B}^{g_1}\mathbf{A}(\boldsymbol{\sigma}_1) + \mathbf{B}\mathbf{B}^{g_1}\mathbf{A}(\boldsymbol{\sigma}_2) & \mathbf{B} \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{A}(\boldsymbol{\sigma}_2) & \mathbf{B} \end{bmatrix} = \mathbf{Q}(\boldsymbol{\sigma}_2)
\end{aligned}$$

We see that the parameters are identifiable iff $\mathbf{A}(\boldsymbol{\sigma}_1)-\mathbf{A}(\boldsymbol{\sigma}_2) = \mathbf{B}\mathbf{B}^{g_1}(\mathbf{A}(\boldsymbol{\sigma}_1) - \mathbf{A}(\boldsymbol{\sigma}_2))$. implies $\boldsymbol{\sigma}_1 = \boldsymbol{\sigma}_2$.

In Theorem 3 we relate this result to the result of the previous section. To this aim we use the following Lemma:

lemma 1. If $\begin{bmatrix}\mathbf{X} & \mathbf{B}\end{bmatrix}$ has full column rank, then

$$\{(\mathbf{I}-\mathbf{B}\mathbf{B}^{g_1})\mathbf{X}\}^{g_1} = \begin{bmatrix}\mathbf{I} \\ -\mathbf{B}^{g_1}\mathbf{X}\end{bmatrix}^{g_1} \begin{bmatrix}\mathbf{X} & \mathbf{B}\end{bmatrix}^{g_1}$$

and $(\mathbf{I}-\mathbf{B}\mathbf{B}^{g_1})\mathbf{X}$ has full column rank.

Proof. First, observe that we can rewrite $(\mathbf{I}-\mathbf{B}\mathbf{B}^{g_1})\mathbf{X}$ as follows:

$$(\mathbf{I}-\mathbf{B}\mathbf{B}^{g_1})\mathbf{X} = \begin{bmatrix}\mathbf{X} & \mathbf{B}\end{bmatrix}\begin{bmatrix}\mathbf{I} \\ -\mathbf{B}^{g_1}\mathbf{X}\end{bmatrix}$$

Second, since $\begin{bmatrix}\mathbf{X} & \mathbf{B}\end{bmatrix}$ has full column rank we obtain the following:

$$\begin{aligned}
\{(\mathbf{I}-\mathbf{B}\mathbf{B}^{g_1})\mathbf{X}\} &= \{(\mathbf{I}-\mathbf{B}\mathbf{B}^{g_1})\mathbf{X}\}\{(\mathbf{I}-\mathbf{B}\mathbf{B}^{g_1})\mathbf{X}\}^{g_1}\{(\mathbf{I}-\mathbf{B}\mathbf{B}^{g_1})\mathbf{X}\} \\
&= \begin{bmatrix}\mathbf{X} & \mathbf{B}\end{bmatrix}\begin{bmatrix}\mathbf{I} \\ -\mathbf{B}^{g_1}\mathbf{X}\end{bmatrix}\begin{bmatrix}\mathbf{I} \\ -\mathbf{B}^{g_1}\mathbf{X}\end{bmatrix}^{g_1}\begin{bmatrix}\mathbf{X} & \mathbf{B}\end{bmatrix}^{g_1}\begin{bmatrix}\mathbf{X} & \mathbf{B}\end{bmatrix}\begin{bmatrix}\mathbf{I} \\ -\mathbf{B}^{g_1}\mathbf{X}\end{bmatrix} \\
&= \begin{bmatrix}\mathbf{X} & \mathbf{B}\end{bmatrix}\begin{bmatrix}\mathbf{I} \\ -\mathbf{B}^{g_1}\mathbf{X}\end{bmatrix}\begin{bmatrix}\mathbf{I} \\ -\mathbf{B}^{g_1}\mathbf{X}\end{bmatrix}^{g_1}\begin{bmatrix}\mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{I} \\ -\mathbf{B}^{g_1}\mathbf{X}\end{bmatrix} \\
&= \begin{bmatrix}\mathbf{X} & \mathbf{B}\end{bmatrix}\begin{bmatrix}\mathbf{I} \\ -\mathbf{B}^{g_1}\mathbf{X}\end{bmatrix}
\end{aligned}$$

which establishes the first part of the Lemma. That $(\mathbf{I}-\mathbf{B}\mathbf{B}^{g_1})\mathbf{X}$ has full column rank follows from the fact that also $\begin{bmatrix} \mathbf{I} \\ -\mathbf{B}^{g_1}\mathbf{X} \end{bmatrix}$ has full column rank:

$$\begin{bmatrix} \mathbf{I} \\ -\mathbf{B}^{g_1}\mathbf{X} \end{bmatrix}^{g_1} \begin{bmatrix} \mathbf{X} & \mathbf{B} \end{bmatrix}^{g_1} \begin{bmatrix} \mathbf{X} & \mathbf{B} \end{bmatrix} \begin{bmatrix} \mathbf{I} \\ -\mathbf{B}^{g_1}\mathbf{X} \end{bmatrix} = \mathbf{I}$$

Theorem 3. If $\begin{bmatrix} \mathbf{A}^*(\boldsymbol{\beta}_1) & \mathbf{B} \end{bmatrix}$ has full column rank then $\mathbf{A}(\boldsymbol{\sigma}_1) - \mathbf{A}(\boldsymbol{\sigma}_2) = \mathbf{B}\mathbf{B}^{g_1}(\mathbf{A}(\boldsymbol{\sigma}_1) - \mathbf{A}(\boldsymbol{\sigma}_2))$ implies $\boldsymbol{\sigma}_1 = \boldsymbol{\sigma}_2$.

Proof. First, observe that $\mathbf{Q}(\boldsymbol{\sigma}_1)\mathbf{Q}(\boldsymbol{\sigma}_1)^{g_1}\mathbf{Q}(\boldsymbol{\sigma}_2)\boldsymbol{\beta}_1 = \mathbf{Q}(\boldsymbol{\sigma}_2)\boldsymbol{\beta}_1$ implies

$$\{\mathbf{A}(\boldsymbol{\sigma}_1) - \mathbf{B}\mathbf{B}^{g_1}\mathbf{A}(\boldsymbol{\sigma}_1) + \mathbf{B}\mathbf{B}^{g_1}\mathbf{A}(\boldsymbol{\sigma}_2)\}\boldsymbol{\beta}_1 = \mathbf{A}(\boldsymbol{\sigma}_2)\boldsymbol{\beta}_1 \quad ,$$

which in turn implies:

$$(\mathbf{I} - \mathbf{B}\mathbf{B}^{g_1})\mathbf{A}(\boldsymbol{\sigma}_1)\boldsymbol{\beta}_1 = (\mathbf{I} - \mathbf{B}\mathbf{B}^{g_1})\mathbf{A}(\boldsymbol{\sigma}_2)\boldsymbol{\beta}_1$$

Using the fact that the MIRID model is bilinear we can rewrite this expression as follows:

$$(\mathbf{I} - \mathbf{B}\mathbf{B}^{g_1})\mathbf{A}^*(\boldsymbol{\beta}_1)\boldsymbol{\sigma}_1 = (\mathbf{I} - \mathbf{B}\mathbf{B}^{g_1})\mathbf{A}^*(\boldsymbol{\beta}_1)\boldsymbol{\sigma}_2$$

This implies $\boldsymbol{\sigma}_1 = \boldsymbol{\sigma}_2$ iff $(\mathbf{I} - \mathbf{B}\mathbf{B}^{g_1})\mathbf{A}^*(\boldsymbol{\beta}_1)$ has full column rank. From Lemma 1 it follows that $(\mathbf{I} - \mathbf{B}\mathbf{B}^{g_1})\mathbf{A}^*(\boldsymbol{\beta}_1)$ has full column rank if $\begin{bmatrix} \mathbf{A}^*(\boldsymbol{\beta}_1) & \mathbf{B} \end{bmatrix}$ has full column rank.

4. The relation between the LLTM and the MIRID model

It is easily seen that the MIRID model is a generalization of the LLTM in the sense that the design matrix of the LLTM, which may not depend on any parameters, is allowed to contain parameters. Here, the reverse situation is considered. We show that the MIRID model can be considered as an LLTM with additional constraints on the parameter space. In other words, we show that the LLTM is a generalization of the MIRID model.

The key observation in showing that the LLTM is a generalization of the MIRID model is that $\boldsymbol{\beta} = \mathbf{Q}(\boldsymbol{\sigma})\boldsymbol{\eta}$ is bilinear. That is, every element of $\boldsymbol{\beta}$ can be written as a bilinear form: $\beta_i = \begin{bmatrix}\boldsymbol{\sigma}^T & 1\end{bmatrix} \mathbf{A}_i\boldsymbol{\eta}$, where $\mathbf{A}_i$ does not depend on the model parameters. These bilinear forms are of a special kind. Specifically, β_i is a sum of terms which are either the product of a parameter from $\boldsymbol{\sigma}$ and a parameter from $\boldsymbol{\eta}$, or a parameter from $\boldsymbol{\eta}$ which is why there is an extra column with entry 1 in $\begin{bmatrix}\boldsymbol{\sigma}^T & 1\end{bmatrix}$. The matrix $\mathbf{A}_i$ determines which product terms enter in the equation for β_i.

It is well known that any bilinear form can be written as a linear form through a reparameterization, a so-called linearization:

$$\beta_i = \begin{bmatrix}\boldsymbol{\sigma}^T & 1\end{bmatrix} \mathbf{A}_i\boldsymbol{\eta} = \mathbf{O}_i'\boldsymbol{\mu} \quad ,$$

where $\mathbf{O}_i$ is a column vector and $\boldsymbol{\mu}$ contains all products of an element from $\begin{bmatrix}\boldsymbol{\sigma}^T & 1\end{bmatrix}$ and an element from $\boldsymbol{\eta}$.

Such a linearization gives an LLTM that encompasses the original MIRID model. The design matrix $\mathbf{O}$ of this LLTM is obtained by stacking the row vectors $\mathbf{O}_i'$ on top of each other. Typically, a number of columns in $\mathbf{OQ}$ will be equal to zero. This means that the corresponding entry in $\boldsymbol{\mu}$ is nowhere involved in the original MIRID model. These columns and the corresponding entries in $\boldsymbol{\mu}$ can be deleted. Furthermore, a number of columns in $\mathbf{O}$ can be the same. This means that the corresponding entries in $\boldsymbol{\mu}$ are only involved in the original MIRID model via their sum. These columns are replaced by only one such column with a corresponding entry in $\boldsymbol{\mu}$ equal to the sum of the entries corresponding to these columns in $\boldsymbol{\mu}$. This process of eliminating zero columns and joining equivalent columns finally gives us the design matrix $\mathbf{Q}_1$ and parameter vector $\boldsymbol{\eta}_1$ of an LLTM that still contains the original MIRID model but typically has far less parameters than the original LLTM. The parameter $\boldsymbol{\eta}_1$ is related to $\boldsymbol{\sigma}$ and $\boldsymbol{\eta}$ in the following way:

$$\boldsymbol{\eta}_1 = \mathbf{Q}_1^{g1}\mathbf{Q}(\boldsymbol{\sigma})\boldsymbol{\eta} \quad . \tag{5}$$

In the section hereafter we illustrate the linearization.

5. Examples

The formulae and results of the previous section are illustrated in the following subsections with some examples.

5.1. *Identifiability*

After a permutation of the rows and columns, the design matrix in Equation 1 can be written as follows:

$$\mathbf{P}_1\mathbf{Q}\mathbf{P}_2 = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{A}(\boldsymbol{\sigma}) & \mathbf{B} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} \sigma_1 & \sigma_2 & 0 & 0 \\ 0 & 0 & \sigma_1 & \sigma_2 \end{bmatrix} & \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{bmatrix} \tag{6}$$

We readily see that the basic parameters $\eta_1, \ldots, \eta_4$ are identifiable from $\boldsymbol{\beta}$: $\eta_1 = \beta_1$, $\eta_2 = \beta_2$, $\eta_3 = \beta_4$, and $\eta_4 = \beta_5$. The identifiability of the remaining parameters is dealt with by reformulating the bilinear system $\begin{bmatrix}\mathbf{A}(\boldsymbol{\sigma}) & \mathbf{B}\end{bmatrix}\boldsymbol{\eta}$ as follows:

$$\begin{bmatrix}\begin{bmatrix}\sigma_1 & \sigma_2 & 0 & 0\\ 0 & 0 & \sigma_1 & \sigma_2\end{bmatrix} & \begin{bmatrix}1\\1\end{bmatrix}\end{bmatrix}\begin{bmatrix}\beta_1\\ \beta_2\\ \beta_4\\ \beta_5\\ \eta_5\end{bmatrix} = \begin{bmatrix}\beta_3\\ \beta_6\end{bmatrix} = \begin{bmatrix}\beta_1 & \beta_2 & 1\\ \beta_4 & \beta_5 & 1\end{bmatrix}\begin{bmatrix}\sigma_1\\ \sigma_2\\ \eta_5\end{bmatrix}$$

Hence, we see that the parameters of this MIRID model are identifiable at points where the matrix $\begin{bmatrix}\beta_1 & \beta_2 & 1\\ \beta_4 & \beta_5 & 1\end{bmatrix}$ has full column rank, which in this case is almost everywhere.

Surprisingly, this MIRID model which has 7 item parameters for 6 items is *not* equivalent to the Rasch model which needs only 6 parameters. This is best seen in the following example:

$$\begin{bmatrix}\beta_3\\ \beta_6\end{bmatrix} = \begin{bmatrix}2\\ 7\end{bmatrix} = \begin{bmatrix}1 & 0 & 1\\ 1 & 0 & 1\end{bmatrix}\begin{bmatrix}\sigma_1\\ \sigma_2\\ \eta_5\end{bmatrix} = \begin{bmatrix}\beta_1 & \beta_2 & 1\\ \beta_4 & \beta_5 & 1\end{bmatrix}\begin{bmatrix}\sigma_1\\ \sigma_2\\ \eta_5\end{bmatrix}$$

which cannot be consistent since it implies that β_3 equals β_6.

5.2. From different item families to equivalent MIRID models

Here we consider a number of different MIRID models that are formally equivalent to the MIRID model in Equation 1.

Let the matrix $\mathbf{P}$ from Theorem 2 be:

$$\mathbf{P} = \begin{bmatrix} \frac{1}{\sigma_1} & -\frac{\sigma_2}{\sigma_1} & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sigma_1} & -\frac{\sigma_2}{\sigma_1} & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -\sigma_1 \end{bmatrix}$$

The determinant of $\mathbf{P}$ equals $-1/\sigma_1$ and hence it is non-singular for σ_1 unequal to zero. With this choice for $\mathbf{P}$ we obtain the following equivalent MIRID model:

$$\mathbf{Q}_1 = \mathbf{QP} = \begin{bmatrix} 1/\sigma_1 & -\sigma_2/\sigma_1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/\sigma_1 & -\sigma_2/\sigma_1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix},$$

where $\boldsymbol{\eta}_1 = \mathbf{Q}_1^{g_1}\mathbf{Q}\boldsymbol{\eta}$. In this example, the first and fourth item are regarded as composite tasks whereas the third and sixth task are now regarded as component tasks. That is, within an item family it is not clear which items are component items and which are composite items.

We now consider a second equivalent model that involves not a different internal structure for the same item families but different item families. The following design matrix:

$$\mathbf{Q}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & \sigma_1 & -\sigma_2 & \sigma_2 & -\sigma_1 \end{bmatrix}$$

together with $\boldsymbol{\eta}_2 = \mathbf{Q}_2^{g_1}\mathbf{Q}\boldsymbol{\eta}$, gives rise to a MIRID model that is equivalent to the model in 1 as can be checked by evaluating the conditions of Theorem 1. Now, the complete set of six items constitutes a single item family consisting of five component items and just one composite item. Moreover, there is no intercept in the regression model which means that the matrix $\mathbf{B}$ has zero columns.

5.3. The MIRID model and the LLTM

In this section we illustrate how for a given MIRID model we can formulate an encompassing LLTM. Consider the following simple MIRID model:

$$\boldsymbol{\beta} = \mathbf{Q}_0(\boldsymbol{\sigma}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \sigma_1 & \sigma_2 & 1 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix} = \begin{bmatrix} \eta_1 \\ \eta_2 \\ \sigma_1\eta_1 + \sigma_2\eta_2 + \eta_3 \end{bmatrix}$$

We easily obtain the following linear form:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma_1\eta_1 \\ \sigma_1\eta_2 \\ \sigma_1\eta_3 \\ \sigma_2\eta_1 \\ \sigma_2\eta_2 \\ \sigma_2\eta_3 \\ \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix} = \begin{bmatrix} \eta_1 \\ \eta_2 \\ \sigma_1\eta_1 + \sigma_2\eta_2 + \eta_3 \end{bmatrix}$$

Deleting zero columns and joining equivalent columns we obtain the encompassing LLTM:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \\ \sigma_1\eta_1 + \sigma_2\eta_2 + \eta_3 \end{bmatrix}$$

which is in fact the Rasch model. This MIRID model is equivalent to the Rasch model since $\mathbf{QQ}^{g_1}$ equals the identity matrix:

$$\mathbf{QQ}^{g_1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \sigma_1 & \sigma_2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\sigma_1 & -\sigma_2 & 1 \end{bmatrix} = \mathbf{I}$$

Another example, in which the encompassing LLTM is not the Rasch model, is the following MIRID model for a facet design from Butter (1994):

$$\boldsymbol{\beta} = \mathbf{Q}(\boldsymbol{\sigma})\boldsymbol{\eta} = \begin{bmatrix} \sigma_1 & 0 & \sigma_2 & 0 & 0 & 1 \\ \sigma_1 & 0 & 0 & \sigma_2 & 0 & 1 \\ \sigma_1 & 0 & 0 & 0 & \sigma_2 & 1 \\ 0 & \sigma_1 & \sigma_2 & 0 & 0 & 1 \\ 0 & \sigma_1 & 0 & \sigma_2 & 0 & 1 \\ 0 & \sigma_1 & 0 & 0 & \sigma_2 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \boldsymbol{\eta} = \begin{bmatrix} \sigma_1\eta_1 + \sigma_2\eta_3 + \eta_6 \\ \sigma_1\eta_1 + \sigma_2\eta_4 + \eta_6 \\ \sigma_1\eta_1 + \sigma_2\eta_5 + \eta_6 \\ \sigma_1\eta_2 + \sigma_2\eta_3 + \eta_6 \\ \sigma_1\eta_2 + \sigma_2\eta_4 + \eta_6 \\ \sigma_1\eta_2 + \sigma_2\eta_5 + \eta_6 \\ \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \\ \eta_5 \end{bmatrix}$$

Through the process for constructing an encompassing LLTM from Section 4 we find the following encompassing LLTM

$$\mathbf{O}_1 = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Since $\mathbf{O}_1$ does not have full column rank, we can in this case delete the first column to produce an equivalent LLTM $\mathbf{O}_2$ which does have full column rank.

The matrix $\mathbf{O}_1$ has an interesting interpretation. The first six items have a facet structure but the components are not related to the last five items as in the MIRID facet model. It is precisely in that sense that this LLTM is encompassing for the MIRID model. This means that we may test the assumption that the facet items indeed involve these subtasks by testing the additional restrictions for the MIRID model. We now consider what these additional restrictions are.

To find the restriction on this LLTM which gives the MIRID model we rewrite $\boldsymbol{\mu}$ as follows:

$$\boldsymbol{\mu} = \mathbf{Q}^{g_1}\mathbf{O}\boldsymbol{\eta} = \begin{bmatrix} -\sigma_1 & \sigma_1 & 0 & 0 & 0 & 0 \\ \sigma_1 & 0 & \sigma_2 & 0 & 0 & 1 \\ \sigma_1 & 0 & 0 & \sigma_2 & 0 & 1 \\ \sigma_1 & 0 & 0 & 0 & \sigma_2 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \\ \eta_5 \\ \eta_6 \end{bmatrix} = \begin{bmatrix} -\sigma_1\eta_1 + \sigma_1\eta_2 \\ \sigma_1\eta_1 + \sigma_2\eta_3 + \eta_6 \\ \sigma_1\eta_1 + \sigma_2\eta_4 + \eta_6 \\ \sigma_1\eta_1 + \sigma_2\eta_5 + \eta_6 \\ \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \\ \eta_5 \end{bmatrix}$$

from which it is immediately clear that $\eta_1 = \mu_5$, $\eta_2 = \mu_6$, $\eta_3 = \mu_7$, $\eta_4 = \mu_8$, and $\eta_5 = \mu_9$. The remaining parameters are determined from the following linear system:

$$\begin{bmatrix} -\sigma_1 & \sigma_1 & 0 & 0 & 0 & 0 \\ \sigma_1 & 0 & \sigma_2 & 0 & 0 & \eta_6 \\ \sigma_1 & 0 & 0 & \sigma_2 & 0 & \eta_6 \\ \sigma_1 & 0 & 0 & 0 & \sigma_2 & \eta_6 \end{bmatrix} \begin{bmatrix} \mu_5 \\ \mu_6 \\ \mu_7 \\ \mu_8 \\ \mu_9 \\ 1 \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{bmatrix}$$

which gives the restriction on $\boldsymbol{\mu}$ that has to be satisfied for the model to be a MIRID model.

6. Discussion and Conclusions

In the context of MIRID models we have found that different componential hypotheses regarding the same items may not be distinguishable. This is a serious problem because from a substantive point of view it is much more interesting to test different componential hypotheses against each other rather than against the Rasch model and it may well turn out that this is not possible. It also means that the concept of item families is ill defined. In other words, it is unclear which items are to be considered as dependent and which as independent variables in the bilinear regression model.

The MIRID model has been presented as a generalization of the LLTM (Butter, 1994; Butter et al., 1998). We have shown that the reverse also holds; that is, for any MIRID model we can construct an encompassing LLTM. This encompassing LLTM may provide a more parsimonious encompassing model than the Rasch model as illustrated in the previous section. Fischer (2002) notes that if a single entry of an LLTM is replaced by a parameter, the resulting MIRID model is again equivalent to a different LLTM. More research is needed however to determine conditions under which the MIRID model and the LLTM are equivalent.

The results of this paper are not only important for the MIRID model but easily carry over to other models in which a bilinear restriction is imposed on the parameter space. In particular when this bilinear restriction has the form in Equation 2. Consider the following example: Together with a set of cognitive items $\mathbf{Y}$ a questionnaire measuring background variables $\mathbf{X}$ is administered. The items in the questionnaire relate to two latent traits τ_1 and τ_2, for instance, test anxiety and need for achievement. In this case it may be reasonable to assume that differences in ability can be explained by differences in the latent traits test anxiety and need for achievement:

$$\begin{bmatrix} \boldsymbol{\theta} \\ \boldsymbol{\tau}_1 \\ \boldsymbol{\tau}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{A}_1(\boldsymbol{\sigma}) & \mathbf{A}_2(\boldsymbol{\sigma}) & \mathbf{B} \\ \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \end{bmatrix} \boldsymbol{\eta}$$

However, it may well turn out that an equivalent model can be constructed in which differences in test anxiety are explained by differences in ability and need for achievement. The problem remains the same if random error is added to the bilinear regression model because the same distribution of $\boldsymbol{\theta}$, $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ can be obtained with different bilinear regression models as long as the distribution of the random error does not depend on $\boldsymbol{\sigma}$ and $\boldsymbol{\eta}$.

References

Bechger, T., Verhelst, N., & Verstralen, H. (2001). Identifiability of nonlinear logistic test models. *Psychometrika, 66*, 357-372.

Bechger, T., Verstralen, H., & Verhelst, N. (2002). Equivalent linear logistic test models. *Psychometrika, 67*, 123-136.

Butter, R. (1994). *Item response model with internal restrictions on item difficulty.* Unpublished doctoral dissertation, KU Leuven.

Butter, R., De Boeck, P., & Verhelst, N. (1998). An item response model with internal restrictions on item difficulty. *Psychometrika, 63*, 47-63.

Fischer, G. (1995). The linear logistic test model. In G. Fischer & I. Molenaar (Eds.), *Rasch models: Foundations, recent developments and applications* (p. 131-155). Berlin, Germany: Springer.

Fischer, G. H. (2002). *Remarks on 'equivalent linear logistic test models' by Bechger, Verstralen, and Verhelst (2002).* (Unpublished Manuscript)

Pringle, R., & Rayner, A. (1971). *Generalized inverse matrices with applications to statistics.* London: Charles Griffin & Co.

Schleiblechner, H. (1972). Das lernen und lösen komplexer denkaufgaben [the learning and solving of complex reasoning items]. *Zeitschrift für experimentelle und angewandte psychologie, 3*, 456-506.