Report Nr 06

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Project 'Optimal Item Selection'

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General Introduction

The purpose of Project 'Optimal Item Selection' is to solve a number of issues in automated test design, making extensive use of optimization techniques. To this end, there has been a close cooperation between the project and, among others, the department of Operations Research at Twente University. In each report, one or several theoretical issues are raised and an attempt is made to solve them. Furthermore, each report is accompanied by one or more computer programs, which are the implementations of the methods that have been investigated. In due time, requests for these programs can be sent to the project director.

T.J.J.M. Theunissen,
project director.

Abstract

A procedure for maximizing the coefficient of generalizability under the constraint of limited resources is presented. The procedure uses optimization techniques that offer an investigator or test constructor the possibility to employ practical constraints. The procedure is illustrated for the two-facet random-model crossed design.

Keywords: generalizability theory, optimization of decision studies, optimization techniques.

Contents

1 Introduction

In generalizability theory (Cronbach, Gleser, Nanda, & Rajaratnam, 1972) a distinction is made between a generalizability study and a decision study. In a generalizability study, accurate estimates of variance components are obtained that can be used in decision studies. Variance components from a generalizability study can, for example, be used to gauge the change in generalizability coefficients when altering the number of conditions for the facets in a decision study using equation (1), given below. In equation (1), the generalizability coefficient, ρ^2, for the two-facet random-model crossed design is expressed as

$$\rho^2 = \frac{\sigma_p^2}{\sigma_p^2 + \frac{\sigma_{p1}^2}{n_1} + \frac{\sigma_{p2}^2}{n_2} + \frac{\sigma_{res}^2}{n_1 n_2}}, \qquad (1)$$

where σ_p^2 is the variance component for persons, σ_{p1}^2 is the variance component for the person by facet 1 interaction, σ_{p2}^2 is the variance component for the person by facet 2 interaction, σ_{res}^2 is the variance component for the person x facet 1 x facet 2 interaction plus error, and n_1 and n_2 are the number of conditions of facet 1 and facet 2. The three interaction variance components constitute the error variance in the two-facet random-model crossed design. The product of the number of conditions of facet 1 and facet 2, $n_1 n_2$, forms the number of observations per subject or other object of measurement. From (1), an investigator or test constructor can not only infer that increasing the number of conditions of the two facets will increase the generalizability coefficient, but also that altering the number of conditions of a facet with a large error variance component will have a different impact on the generalizability coefficient than altering the number of conditions of a facet with a small error variance component.

To improve the quality and economy of decision studies, Woodward and Joe (1973) developed two procedures. One can be used to solve the problem of determining the minimum number of observations for a specified generalizability coefficient. Recently, a more versatile extension of their procedure for this problem was presented by Sanders, Theunissen, and Baas (1989). In response to discussions by Cronbach et al. (1972, pp. 173, 182), Woodward and Joe (1973) also developed a procedure for the problem of maximizing the generalizability coefficient subject to the constraint that the total number of observations is fixed. Because this constraint implies that the investigator is free to vary the number of conditions of the facets and resources required per condition of each facet do not differ, their procedure does not correspond to the application of decision studies in practice. Investigators do not conduct decision studies under the constraint of a fixed total number of observations but under other, different constraints, with the most important being an upper limit on resources - including monetary, time, and expertise. Therefore, resources constitute a factor that has to be considered when decisions about the composition of measurement instruments are made. However, despite its relevance, procedures that could help investigators to make these decisions are lacking.

The problem of maximizing the coefficient of generalizability under the constraint of an upper limit on resources is addressed in this paper. The problem is first formulated as an optimization problem. A two-step procedure for solving the optimization problem for the two-facet random-model crossed design is presented, and an example is used to illustrate the procedure.

2 Optimization Problem

Maximizing the generalizability coefficient of the two-facet random-model crossed design is equivalent to minimizing the error variance.
Denoting σ^2_{p1}, σ^2_{p2}, and σ^2_{p12} by vc_1, vc_2, and vc_{12} respectively, the objective-function for this optimization problem is formulated as

$$\text{minimize} \qquad \frac{vc_1}{n_1} + \frac{vc_2}{n_2} + \frac{vc_{12}}{n_1 n_2} \quad . \qquad (2)$$

The minimization statement (2) refers to the value of the objective-function that results when different numbers of conditions, n_1 and n_2, are used for facet 1 and 2.

A complete description of our optimization problem includes two other constraints. First, the constraint specifying the resources available for the decision study:

$$c_1 n_1 + c_2 n_2 + c_{12} n_1 n_2 \leq r \quad . \qquad (3)$$

In (3), the resources required by the conditions of facet 1 are specified by the term $c_1 n_1$, with c_1 being the 'cost' of one condition of facet 1. The resources required by the conditions of facet 2 are specified by the term $c_2 n_2$, with c_2 being the cost of one condition of facet 2. The number of observations per subject or other object of measurement equals $n_1 n_2$. Thus, denoting the cost of one observation for the sample of subjects to be tested by c_{12}, the resources necessary for the total number of observations are specified by the term $c_{12} n_1 n_2$. The right-hand term of (3) specifies an upper limit on the resources available for the decision study. Here, it is assumed that the cost per condition does not vary within the same facet but can vary across different facets, and that the cost of conditions for all the facets are expressed in the same units as r. Second, because feasible values for n_1 and n_2 have to be integer values and each facet in a two-facet design has to have at least one condition, a lower bound integer constraint has to be included as well:

$$n_1 \text{ and } n_2 \text{ integer} \geq 1 \; . \tag{4}$$

In addition to (3) and (4), other constraints can be employed, as discussed later.

3 Solution for the Two-Facet Design Problem

Optimal integer solutions for n_1 and n_2 of the optimization problem defined by (2), (3), and (4) are obtained in two steps. In the first, solutions are derived for a continuous relaxation of constraint (4), that is, $n_1, n_2 \geq 0$. In the second step these optimal continuous solutions are used as the bounds in a branch-and-bound procedure (see Papadimitriou & Steiglitz, 1985, p. 443) to obtain the optimal integer solutions.

Continuous Solution

By defining $\frac{1}{n_1} = x_1 + \frac{c_1}{r}$, $\frac{1}{n_2} = x_2 + \frac{c_2}{r}$, $v_{12} = vc_{12}$,

$v_1 = vc_1 + \frac{vc_{12}c_2}{r}$, and $v_2 = vc_2 + \frac{vc_{12}c_1}{r}$, a transformed version of

the problem can be formulated as:

$$\text{minimize} \quad v_1x_1 + v_2x_2 + v_{12}x_1x_2 + \frac{vc_1c_1}{r} + \frac{vc_2c_2}{r} + \frac{vc_{12}c_1c_2}{r^2} \tag{5}$$

$$\text{subject to} \quad x_1x_2 \geq \frac{c_{12}}{r} + \frac{c_1c_2}{r^2}, \tag{6}$$

$$-\frac{c_1}{r} \leq x_1 \leq 1 - \frac{c_1}{r}, \text{ and} \tag{7}$$

$$-\frac{c_2}{r} \leq x_2 \leq 1 - \frac{c_2}{r}.$$

However, $x_1 = \frac{1}{n_1} - \frac{c_1}{r} < 0$ or $x_2 = \frac{1}{n_2} - \frac{c_2}{r} < 0$ would imply $n_1c_1 > r$ or $n_2c_2 > r$, which is inconsistent with the resources constraint (3).

Hence (7) is modified to $0 \leq x_1 \leq 1 - \frac{c_1}{r}$, and $0 \leq x_2 \leq 1 - \frac{c_2}{r}$, thus

leading to bounds on n_1 and n_2 of $\left[1, \frac{r}{c_1}\right]$ and $\left[1, \frac{r}{c_2}\right]$.
The formulation of the problem is further simplified by defining the right-hand terms of (6) and (7) as:

$$b = \frac{c_{12}}{r} + \frac{c_1 c_2}{r^2}, \quad u_1 = 1 - \frac{c_1}{r}, \text{ and } \quad u_2 = 1 - \frac{c_2}{r},$$

so that the continuous optimization problem can be expressed as:

$$\text{minimize} \quad v_1 x_1 + v_2 x_2 + v_{12} x_1 x_2 + \text{constant} \tag{8}$$

$$\text{subject to} \quad x_1 x_2 \geq b, \tag{9}$$

$$0 \leq x_1 \leq u_1, \text{ and} \tag{10}$$

$$0 \leq x_2 \leq u_2$$

Note that the last three terms of (5) are denoted as a constant in (8). The above optimization problem can be solved by standard methods from non-linear optimization using Kuhn-Tucker necessary conditions. The Lagrange function is given by

$$L(x,\lambda) = v_1 x_1 + v_2 x_2 + v_{12} x_1 x_2 - \lambda_1 (x_1 x_2 - b) + \lambda_2 (x_1 - u_1)$$
$$- \lambda_3 x_1 + \lambda_4 (x_2 - u_2) - \lambda_5 x_2 .$$

To guarantee a non-empty solution set (see discussion later on), the assumption $u_1 u_2 \geq b$ or its equivalent $c_1 + c_2 + c_{12} \leq r$ has to be made. Since a so-called constraint qualification holds with this assumption, the Lagrange function results in the following system of Kuhn-Tucker necessary conditions:

$$\frac{\partial L}{\partial x_1} = v_1 + v_{12} x_2 - \lambda_1 x_2 + \lambda_2 - \lambda_3 = 0 \tag{11}$$

$$\frac{\partial L}{\partial x_2} = v_2 + v_{12} x_1 - \lambda_1 x_1 + \lambda_4 - \lambda_5 = 0 \tag{12}$$

$$\lambda_1(x_1x_2 - b) = 0 \quad (13)$$

$$\lambda_2(x_1 - u_1) = 0 \quad (14)$$

$$\lambda_3 x_1 = 0 \quad (15)$$

$$\lambda_4(x_2 - u_2) = 0 \quad (16)$$

$$\lambda_5 x_2 = 0 \quad (17)$$

$$\lambda_1,\ \lambda_2,\ \lambda_3,\ \lambda_4,\ \lambda_5 \geq 0 \quad (18)$$

$$x_1x_2 \geq b \quad (19)$$

$$0 \leq x_1 \leq u_1,\ 0 \leq x_2 \leq u_2 \quad (20)$$

These conditions are necessary but not sufficient to produce a solution for the continuous problem being considered here. However, since the feasible region defined by conditions (9) and (10) is closed and bounded in R^2, and the objective function is continuous there, the existence of a solution is guaranteed. Moreover, any solution to the constrained minimization problem is included in the solution set for the Kuhn-Tucker conditions. This solution set will be derived next, with a candidate solution denoted by x_1^* and x_2^*.

If either $x_1 = 0$ or $x_2 = 0$, condition (19) cannot be fulfilled since $b > 0$. Hence, in a feasible solution both $x_1 > 0$ and $x_2 > 0$ must hold, from which, because of condition (15) and (17), it follows that $\lambda_3 = \lambda_5 = 0$. Since $\lambda_2 \geq 0$ and $\lambda_4 \geq 0$, it is concluded that $\lambda_1 > 0$, otherwise conditions (11) and (12) cannot have a solution. With $\lambda_1 > 0$, condition (13) implies that $x_1x_2 = b$ must hold.

A solution with $x_1^* = u_1$ and $x_2^* < u_2$, given $x_1^*x_2^* = b$, can only satisfy the Kuhn-Tucker conditions if $\frac{b}{u_1} < u_2$. Substituting these values in (11) and (12) with $\lambda_3 = \lambda_5 = 0$, as established above, and $\lambda_4 = 0$, because of (16), results in the following system of two equations in the two variables

λ_1 and λ_2: $v_1 + v_{12}\frac{b}{u_1} - \lambda_1\frac{b}{u_1} + \lambda_2 = 0$, and $v_2 + v_{12}u_1 - \lambda_1 u_1 = 0$.

Solving this system gives $\lambda_1 = \frac{v_2 + v_{12}u_1}{u_1}$, and $\lambda_2 = \frac{bv_2}{u_1^2} - v_1$. Since, because of (18), $\lambda_2 \geq 0$, a solution $x_1^* = u_1$, $x_2^* = \frac{b}{u_1} < u_2$ exists, if and only if $v_1 \leq \frac{bv_2}{u_1^2}$. Similarly, if $v_2 \leq \frac{bv_1}{u_2^2}$, a solution $x_1^* = \frac{b}{u_2} < u_1$, $x_2^* = u_2$ results.

A solution with $x_1^* = u_1$, $x_2^* = u_2$ ($\equiv n_1 = n_2 = 1$) can only exist if $u_1u_2 = b$ ($\equiv c_1 + c_2 + c_{12} = r$), which has already been excluded from consideration.

What remains to be derived is the solution for the general case with $x_1^* < u_1$ and $x_2^* < u_2$, for which $\lambda_2 = \lambda_4 = 0$ according to (14) and (16). From (11), with $\lambda_2 = 0$, $x_2^* = \frac{v_1}{\lambda_1 - v_{12}}$; and from (12), with $\lambda_4 = 0$,

$$x_1^* = \frac{v_2}{\lambda_1 - v_{12}} .$$

These two equations give $x_2^* = \frac{v_1}{v_2} x_1^*$. Substitution in (9) produces

$x_1^{*2} = b\,\frac{v_2}{v_1}$, which in turn yields $x_1^* = \left(\frac{bv_2}{v_1}\right)^{1/2}$, and $x_2^* = \left(\frac{bv_1}{v_2}\right)^{1/2}$.

It is easily verified that the four possible solutions exclude each other. For example, for the solution of the general case, solution $x_1^* = \left(\frac{bv_2}{v_1}\right)^{1/2} < u_1$ and $x_2^* = \left(\frac{bv_1}{v_2}\right)^{1/2} < u_2$ imply that the inequalities

$v_1 > \frac{bv_2}{u_1^2}$ and $v_2 > \frac{bv_1}{u_2^2}$ must hold, which excludes as possible solutions $x_1^* = u_1$ and $x_2^* = u_2$. The four possible solutions are listed in Table 1.

Table 1 Four possible solutions for the continuous optimization problem

1. $u_1u_2 = b$	$x_1^* = u_1$	$\Rightarrow$	$n_1^* = 1$
	$x_2^* = u_2$	$\Rightarrow$	$n_2^* = 1$
2. $u_1u_2 > b$	$x_1^* = u_1$	$\Rightarrow$	$n_1^* = 1$
$v_1 \le \dfrac{bv_2}{u_1^2}$	$x_2^* = \dfrac{b}{u_1}$	$\Rightarrow$	$n_2^* = \left[\dfrac{rc_{12} + c_1c_2}{r(r-c_1)} + \dfrac{c_2}{r}\right]^{-1}$
3. $u_1u_2 > b$	$x_1^* = \dfrac{b}{u_2}$	$\Rightarrow$	$n_1^* = \left[\dfrac{rc_{12} + c_1c_2}{r(r-c_2)} + \dfrac{c_1}{r}\right]^{-1}$
$v_1 \le \dfrac{bv_2}{u_2^2}$	$x_2^* = u_2$	$\Rightarrow$	$n_2^* = 1$
4. $u_1u_2 > b$	$x_1^* = \left[\dfrac{bv_2}{v_1}\right]^{\frac{1}{2}}$	$\Rightarrow$	$n_1^* = \left[\left(\dfrac{v_2(rc_{12} + c_1c_2)}{v_1r^2}\right)^{\frac{1}{2}} + \dfrac{c_1}{r}\right]^{-1}$
$v_1 > \dfrac{bv_2}{u_1^2}$, $v_2 > \dfrac{bv_1}{u_2^2}$	$x_2^* = \left[\dfrac{bv_1}{v_2}\right]^{\frac{1}{2}}$	$\Rightarrow$	$n_2^* = \left[\left(\dfrac{v_1(rc_{12} + c_1c_2)}{v_2r^2}\right)^{\frac{1}{2}} + \dfrac{c_2}{r}\right]^{-1}$

The last column of this table contains the solutions after back-transformation. Because solutions 2 and 3 with n_1^* or n_2^* equal to 1 are not likely to occur, the solution for the general case, solution 4, can be regarded as the appropriate optimal continuous solution for the two-facet optimization problem.

Integer Solution

The optimal continuous solution for the general case, derived in the preceding section, will be used as a starting point for a branch-and-bound procedure to obtain the optimal integer solution for the problem. Assume that n_1^* and n_2^* are the optimal continuous solutions, and that not both values are integer. Let $\lfloor n_1^* \rfloor$ denote that n_1^* is rounded down, and $\lceil n_1^* \rceil$ that n_1^* is rounded up, then two problems have to be solved excluding the value n_1^*, but maintaining feasibility for all possible integer values of n_1. The first problem is

$$\text{minimize} \quad \frac{\sigma^2_{p1}}{n_1} + \frac{\sigma^2_{p2}}{n_2} + \frac{\sigma^2_{res}}{n_1 n_2}$$

$$\text{subject to} \quad c_1 n_1 + c_2 n_2 + c_{12} n_1 n_2 \leq r \ ,$$

$$n_1, n_2 \geq 1 \ , \text{ and}$$

$$n_1 \leq \lfloor n_1^* \rfloor \ .$$

In a solution $(\hat{n}_1, \hat{n}_2)$ for this problem, $\hat{n}_1 = \lfloor n_1^* \rfloor$ must hold, because if $n_1^* \neq \lfloor \hat{n}_1 \rfloor$ the rounding down constraint would be irrelevant, resulting in the non-integer solution to the original problem. As before, the resources constraint has to be satisfied as well. Consequently, either the solution for the first problem is $\hat{n}_1 = \lfloor n_1^* \rfloor$ and, from the resources constraint,

$\hat{n}_2 = \dfrac{r - c_1 \hat{n}_1}{c_2 + c_{12} \hat{n}_1}$, or $\hat{n}_1 = 1$ and $\hat{n}_2 = \dfrac{r - c_1}{c_2 + c_{12}}$. The second problem differs from the first in that the rounding down constraint is replaced by the rounding up constraint $n_1 \geq \lceil n_1^* \rceil$. The solution for the second problem is $\hat{n}_1 = \lceil n_1^* \rceil$, and $\hat{n}_2 = \dfrac{r - c_1\hat{n}_1}{c_2 + c_{12}\hat{n}_1}$.

Each of these two problems gives rise to two new problems, with additional constraints regarding n_2 whenever the value $\hat{n}_2$ is non-integer. If, for instance, $\hat{n}_1$ is non-integer in the first problem, the constraint sets for the following two problems are given by (1) $n_2 \geq 1$, $1 \leq n_1 \leq \lfloor \hat{n}_1 \rfloor$, and (2) $n_2 \geq 1$, $n_1 \geq \lceil \hat{n}_1 \rceil$. Branching continues until either an integer solution or an infeasible constraint set is found. Each time an integer solution is obtained during the branching process, its value is compared with that of the previous best solution and accepted as the new best solution or rejected as such. At the end of the process the current best solution is the optimal integer solution.

4 Example

Our procedure will be illustrated for the two-facet random-model crossed design using the example from Woodward and Joe (1973, p. 179) with the following variance components: $\hat{\sigma}^2_p = 5.435$, $\hat{\sigma}^2_{p1} = 3.421$, $\hat{\sigma}^2_{p2} = 1.140$, and $\hat{\sigma}^2_{res} = 11.850$. The objective-function for this example is

$$\text{minimize} \qquad \frac{3.421}{n_1} + \frac{1.140}{n_2} + \frac{11.850}{n_1 n_2} \, .$$

Assuming that a condition of facet 1 (e.g., essay questions), costs 40 dollars, a condition of facet 2 (e.g., raters), costs nothing and one observation (i.e., the answers of all examinees to one essay question rated by one rater), costs 80 dollars, and that the budget for the decision study is limited to 3000 dollars, the resources constraint for this example can be stated as

$$40n_1 + 80n_1n_2 \leq 3000.$$

The optimal integer solutions $\hat{n}_1$ and $\hat{n}_2$ for this optimization problem are derived in two steps. First, using solution 4 from Table 1, the optimal continuous solutions $n_1^* = 8.8$ and $n_2^* = 3.8$ are obtained. Second, because both solutions are non-integer, a branch-and-bound procedure is needed to find the optimal integer solutions. For the problem with additional constraint $n_1 \leq \lfloor n_1^* \rfloor = 8$, one finds a solution $\hat{n}_1 = 8$, $\hat{n}_2 = 4.2$ and for the problem with $n_1 \geq \lceil n_1^* \rceil = 9$, a solution $\hat{n}_1 = 9$, $\hat{n}_2 = 3.7$. Further branching gives rise to four problems with constraint sets including $n_1, n_2 \geq 1$ and: (1) $n_1 \leq 8$, $n_2 \leq 4$; (2) $n_1 \leq 8$, $n_2 \geq 5$; (3) $n_1 \geq 9$, $n_2 \leq 3$, and (4) $n_1 \geq 9$, $n_2 \geq 4$. The search process for the example with resources limited to 3000 dollars is shown in Figure 1.

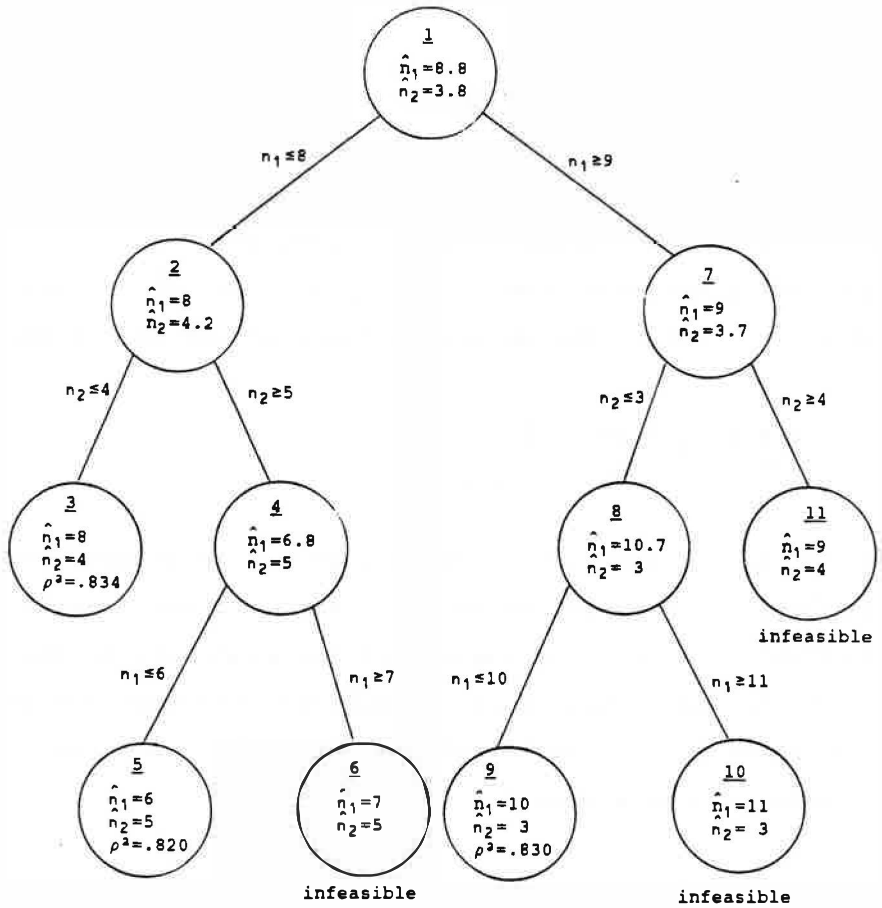

Figure 1   Search-tree for the two-facet example

Starting from the optimal continuous solution in node 1, the strategy to traverse the search-tree is depth-first and from left to right, as indicated by the numbering of the nodes. Node 3 produces the first candidate solution, which is not improved by the solution found in node 5. The solution in node 6 does not satify the budget constraint and is therefore an infeasible solution. Other infeasible solutions are found in node 10 and 11. The solution found in node 9 is slightly worse than the solution in node 3 and thus rejected. From the search-process in Figure 1 as well other solutions presented in Table 2, it can be concluded that a more exhaustive search will result in either infeasible solutions or solutions that do not improve the solution in node 3. Therefore, the solution in node 3, $(\hat{n}_1, \hat{n}_2) = (8, 4)$, that is, eight essay questions of which the answers of all examinees have to be rated by four raters, is the optimal integer solution for this problem.

Table 2 Values of n_1, n_2, variance components, ρ^2 and r, when c_1 = 40 dollars, c_2 = 0 dollars and c_{12} = 80 dollars

n_1	$c_1 n_1$	n_2	$n_1 n_2$	$c_{12} n_1 n_2$	$\hat{\sigma}^2_p$	$\frac{\hat{\sigma}^2_{p1}}{n_1}$	$\frac{\hat{\sigma}^2_{p2}}{n_2}$	$\frac{\hat{\sigma}^2_{res}}{n_1 n_2}$	ρ^2	r
6	240	4	24	1920	5.435	.570	.285	.494	.802	2160
6	240	5	30	2400	5.435	.570	.228	.395	.820	2640
7	280	3	21	1680	5.435	.489	.380	.564	.791	1960
7	280	4	28	2240	5.435	.489	.285	.423	.820	2520
7	280	5	35	2800	5.435	.489	.228	.339	.837	3080
8	320	3	24	1920	5.435	.428	.380	.494	.807	2240
8	320	4	32	2560	5.435	.428	.285	.370	.834	2880
9	360	3	27	2160	5.435	.380	.380	.439	.819	2520
9	360	4	36	2880	5.435	.380	.285	.329	.845	3240
10	400	3	30	2400	5.435	.342	.380	.395	.830	2800
10	400	4	40	3200	5.435	.342	.285	.296	.855	3600
11	440	3	33	2640	5.435	.311	.380	.359	.838	3080
12	480	2	24	1920	5.435	.285	.570	.494	.801	2400

Although the resources constraint employed here is not based on real data, the example is realistic since in most decision studies, differences between the number of observations will have more impact on the necessary resources than on the generalizability coefficient. A small difference between generalizability coefficients will therefore correspond with a large difference in resources (for example, compare solution $(\hat{n}_1, \hat{n}_2) = (7, 3)$ with solution $(\hat{n}_1, \hat{n}_2) = (8, 3)$).

The present example is also used to demonstrate that our procedure is a generalization of Woodward and Joe's. Their procedure is a simplification of the resources constraint (9) employed by our procedure, obtained by substituting the left-hand term of (9) with the number of observations, the 'greater than or equal to' sign with the 'equal' sign and the right-hand term with a specific number of observations. Then, using solution 4 from Table 1, the optimal continuous number of conditions of facet 1 and facet 2 under the constraint of a fixed number of observations is given by

$$n_1^* = \left[\frac{b^{-1} v_1}{v_2}\right]^{\frac{1}{2}}, \text{ and } \quad n_2^* = \left[\frac{b^{-1} v_2}{v_1}\right]^{\frac{1}{2}} .$$

Substituting b^{-1}, the number of observations specified by the investigator, with the term L, leads to Woodward and Joe's equations

(1973, p. 176). According to the preceding equations, the optimal continuous number of conditions of facet 1 and facet 2, with L is equal to 400, are $n_1^* = 34.65$ and $n_2^* = 11.55$. Rounding off these non-integer values, their solution ($\hat{n}_1 = 40$, $\hat{n}_2 = 10$) with a generalizability coefficient equal to .9595, is the optimal integer solution under the constraint of 400 observations. However, a generalizability coefficient of .9596 can be obtained with a smaller number of observations, explicitly, $\hat{n}_1 = 36$ and $\hat{n}_2 = 11$. This solution would have been obtained with our two-step procedure, and it can be proven that this solution is indeed the optimal integer solution (Sanders et al., 1989). Although the differences in generalizability coefficients are trivial, this solution is to be preferred over solutions ($\hat{n}_1 = 33$, $\hat{n}_2 = 12$) and ($\hat{n}_1 = 35$, $\hat{n}_2 = 11$) with respectively 396 and 385 observations. The latter procedure also shows the insensitivity of higher values of generalizability coefficients to even major changes in the design. Woodward and Joe's solution is less parsimonious than ours because their procedure does not allow for the number of observations to vary. With L set to 396, their solution would have been the same as ours. Besides the efficiency resulting from defining the objective-function as an inequality constraint, our procedure also allows to handle additional constraints much more easily. These two features make it a very versatile procedure.

As was shown by the example above, it is important to note that when the resources constraint is replaced by a constraint specifying the number observations, $v_1 = vc_1$ and $v_2 = vc_2$ in the above expressions. For resources constraints comparable to the budget constraint employed in our example, the ratio v_1/v_2 will roughly approximate the ratio vc_1/vc_2. This means that for two-facet random-model designs, the optimal number of conditions for facet 1 and facet 2 will be proportional to the ratio of the error variance components.

5 Conclusions and Discussion

The procedure proposed in this article enables an investigator or test constructor to conduct a decision study under the constraint of limited resources. Although in practice resources will be the most important constraint, the procedure admits the possibility of others as well. For example, an investigator who wishes a measurement instrument composed of at least six times as many conditions of facet 1 (e.g., essay questions) than of facet 2 (e.g., raters) can achieve this by employing the constraint $n_1 \geq 6n_2$. Adding this constraint to the present example would result in the optimal solution ($\hat{n}_1 = 12$, $\hat{n}_2 = 2$) = 2400 dollars, as can be seen in Table 2. The possibility of including additional constraints makes the procedure particularly useful for solving many practical measurement construction problems.

Because of the consequences for the measurement instruments attached to the choice of constraints, they should be the result of a deliberate choice and carefully examined. Following Sanders et al. (1989), a distinction can be made between psychometric constraints and other, economic or practical, constraints. A psychometric constraint is one in accord with the psychometric structure of the problem - the magnitude of the error variance components. The inequality $n_1 \geq n_2$, for instance, would be a psychometric constraint for the present example, because a higher generalizability coefficient will be obtained by increasing the number of conditions of facet 1 than by increasing the number of conditions of facet 2. On the other hand, the constraint $n_1 \leq n_2$ would not be considered as such. Another useful distinction to be made is between equality and inequality constraints. Inequality constraints are to be preferred over equality constraints, because equality constraints often lead to less efficient solutions. The latter was illustrated by Woodward and Joe's equality constraint of a fixed number of observations. It should be emphasized that dependent on the employment of psychometric and/or other constraints, quite different optimal solutions can be obtained. Obviously, employing only psychometric constraints will lead to higher generalizability coefficients than employing other constraints. An investigator who considers the generalizability coefficient obtained inadequate, can take two decisions. The first is to change the resources constraint (e.g., increasing the budget). The second is to change the design of the decision study (e.g., nesting or deleting a facet).

An optimization model for classical test theory has recently been presented by Van der Linden and Adema (1988). Assuming an item bank with estimates of item parameters based on classical test theory (i.e., item difficulty and item test correlation), their procedure maximizes coefficient alpha permitting as many practical constraints as test constructors can think of. Their procedure can be viewed as an extension of our procedure for one-facet designs. Instead of a (stratified) random selection of items from the item bank, their procedure makes a selection of specific items based on the psychometric properties of these items. For tests of usual length, however, the differences between reliability coefficients constructed by the two procedures can be expected to be small and of little practical relevance. For one-facet designs, the aspect of diminishing returns should also be taken into account, because higher values of reliability coefficients are hardly influenced by changes in the number of items.

Two extensions of the procedure presented here are obvious. First, the assumption of constant cost per condition within the same facet can be relaxed to allow for varying costs for conditions within the same facet. Second, the procedure can be generalized to multifacet and nested designs, and designs for mixed models. Although the mathematics of these generalizations is rather tedious, they are quite straightforward as demonstrated for a similar procedure by Sanders et al. (1989).

Applications of integer optimization techniques for test design within the framework of latent trait theory have been presented by Theunissen (1985), and Van der Linden and Boekkooi-Timminga (1989). To make these techniques accessible to the practitioner, a computer program has been developed that can handle a variety of test designs and constraints. Information about this program, called 'Optimal item selection', can be obtained from the second author of the current paper. Unfortunately, present computer programs for performing generalizability and decision studies (see Crick & Brennan, 1982; Brennan, 1983; Cardinet & Tourneur, 1985) do not use optimization techniques. As far as we know, only one experimental program applying these techniques in generalizability theory has been developed (see Sanders et al., 1989). Using as input estimates of variance components, this program enables an investigator to specify an acceptable threshold for a generalizability coefficient and employ constraints to obtain the optimal number of conditions of facets for a limited number of designs. What is needed, however, is a general program that can handle various kinds of objective-functions, designs and constraints.

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