Measurement and Research Department Reports 91-9

Alternative Solutions For Optimization Problems In Generalizability Theory

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Arnhem, 1991

Abstract

Solutions for the problem of maximizing the generalizability coefficient under a budget constraint are presented. It is shown that the Cauchy-Schwarz inequality can be applied to derive optimal continuous solutions for the number of conditions of each facet.

Key words: generalizability theory, Cauchy-Schwarz inequality, optimal designs.

The author thanks Sjoerd Baas and Agnes Broeren for their many helpful remarks. A shorter version of this manuscript will be published by Psychometrika.

Introduction

A procedure for the problem of maximizing the generalizability coefficient of the two-facet random-model crossed design under a budget constraint was recently presented by Sanders, Theunissen, and Baas (1991), who formulated this optimization problem as:

$$\text{minimize} \qquad \frac{\sigma^2_{p1}}{n_1} + \frac{\sigma^2_{p2}}{n_2} + \frac{\sigma^2_{res}}{n_1 n_2} = V, \tag{1}$$

$$\text{subject to} \qquad C_1 n_1 + C_2 n_2 + C_{12} n_1 n_2 \le C, \text{ and} \tag{2}$$

$$n_1 \text{ and } n_2 \text{ integer} \ge 1. \tag{3}$$

Maximizing the generalizability coefficient is equivalent to minimizing the error variance, that is, the sum of the three interaction variance components. In (1), σ^2_{p1} is the variance component for the person by Facet 1 interaction, σ^2_{p2} is the variance component for the person by Facet 2 interaction, σ^2_{res} is the variance component for the person x Facet 1 x Facet 2 interaction plus error, and n_1 and n_2 are the number of conditions of Facet 1 and Facet 2. The minimization statement in (1) expresses that the value of the objective-function is determined by the numbers of conditions, n_1 and n_2, used for Facet 1 and 2.

In the cost function in (2), the right-hand term specifies an upper limit on the budget. The costs required by the conditions of Facet 1 are specified by the term $C_1 n_1$, C_1 being the cost of one condition of Facet 1. The costs required by the conditions of Facet 2 are specified by the term $C_2 n_2$, C_2 being the cost of one condition of Facet 2. The number of observations per subject equals $n_1 n_2$. Thus, denoting the cost of one observation for the sample of subjects to be tested by C_{12}, the costs necessary for the total number of observations are specified by the term $C_1 n_1 n_2$.

The lower bound integer constraint (3) states that feasible values for n_1 and n_2 have to be integer values and each facet has to have at least one condition.

In the two-step procedure proposed by Sanders et al. (1991), optimal continuous solutions for n_1 and n_2 are derived for a continuous relaxation of (3) in the first step. These continuous solutions are then used as the bounds in a branch-and-bound algorithm to obtain the optimal integer solutions. The use of the Lagrange multipliers method to attain the optimal continuous solutions, however, can result in complex derivations. But, as first shown by Stuart (1954) for problems of sample survey theory, derivations of optimum solutions can be simplified if the Cauchy-Schwarz inequality applies. This inequality states that for any sets of real numbers

$\{a_h\}$, $\{b_h\}$, $(h = 1, 2, \ldots, p)$,

$$(\Sigma a_h^2)(\Sigma b_h^2) \geq (\Sigma a_h b_h)^2, \tag{4}$$

with equality occurring if and only if $a_h/b_h = k$ for all h and some constant k. According to Stuart (1954, p. 239), the sampling variance in most of the problems in sample survey theory takes the form $\Sigma \frac{v_h}{n_h} = v$, where v_h is the function of population parameters only, and n_h is a function of sample numbers only. The cost functions generally considered are of the form $\Sigma n_h c_h = c$, where the c_h are fixed cost constants. From (4) it therefore follows that

$$vc = \left(\Sigma \frac{v_h}{n_h}\right)(\Sigma n_h c_h) \geq \left(\Sigma (v_h c_h)^{1/2}\right)^2. \tag{5}$$

Since the right-hand side of (5) is independent of the n_h, the minimization of vc for fixed c (or for fixed v) and variation in the n_h is achieved when, from (4) and (5),

$$\frac{n_h^2 c_h}{v_h} = k^2 \text{ (all } h\text{)}. \tag{6}$$

In the survey sampling literature (e.g., Cochran, 1977), applications of the Cauchy-Schwarz inequality to optimization problems abound. The purpose of the present note is to show its applicability to optimization problems recurring in generalizability theory.

Solutions for Two-Facet Optimization Problems

Other linear cost functions besides (2) can arise if C_1, C_2 or C_{12} is equal to zero. Functions with C_1 or C_2 and C_{12} equal to zero, however, lead to trivial optimization problems. Moreover, the cost function with C_1 and C_2 equal to zero, that is, cost function $\frac{C}{C_{12}} \le n_1 n_2$, reduces the problem of optimum allocation for fixed cost to the problem of optimum allocation for a fixed number of observations. By replacing the inequality constraint in the cost function by an equality constraint, the optimal continuous solutions for this optimization problem,

$$n_1 = \left(\frac{\sigma^2_{p1}}{\sigma^2_{p2}} \cdot \frac{C}{C_{12}} \right)^{½}, \text{ and } n_2 = \left(\frac{\sigma^2_{p2}}{\sigma^2_{p1}} \cdot \frac{C}{C_{12}} \right)^{½},$$

were derived by Woodward and Joe (1973). Therefore, three other potential linear cost functions can be distinguished:

(7) $C_1 n_1 + C_{12} n_1 n_2 \le C$, (8) $C_2 n_2 + C_{12} n_1 n_2 \le C$, and (9) $C_1 n_1 + C_2 n_2 \le C$.

These three cost functions and (1) result in three different optimization problems.

The solution strategy for each of the three optimization problems is as follows. First, the inequality constraint of the cost function is replaced by an equality constraint. Second, the terms in the objective-function and the cost function are reformulated so that (5) applies. For example, for the optimization problem defined by (1) and (7), we note that the cost function does not include a linear term with n_2. Dividing (7) by $n_1 n_2$, however, we can rewrite (7) as $C_{12} = -\frac{C_1}{n_2} + \frac{C}{n_1 n_2}$. Moreover, by adding a constant to (1) and multiplying this constant by C_{12}, (1) can be expressed as

$$\frac{\sigma^2_{p1}}{n_1} + \frac{\sigma^2_{p2}}{n_2} + \frac{\sigma^2_{res}}{n_1 n_2} + \frac{\sigma^2_{p2}}{C_1} \cdot C_{12} =$$

$$\frac{\sigma^2_{p1}}{n_1} + \frac{\sigma^2_{p2}}{n_2} + \frac{\sigma^2_{res}}{n_1 n_2} + \left(-\frac{\sigma^2_{p2}}{n_2} + \frac{\sigma^2_{p2} C}{C_1 n_1 n_2} \right).$$

From (4) and (5) we obtain

$$\frac{\sigma_{p1}^2}{n_1} : c_1 n_1 = k^2 \text{ , and} \tag{10}$$

$$\frac{\left(\sigma_{res}^2 + \sigma_{p2}^2 \cdot \dfrac{c}{c_1}\right)}{n_1 n_2} : c_{12} n_1 n_2 = k^2. \tag{11}$$

Multiplying (10) by $(c_1)^{1/2}$ and (11) by $(c_{12})^{1/2}$ results in

$$(\sigma_{p1}^2)^{1/2}(c_1)^{1/2} = k\, c_1 n_1, \text{ and}$$

$$\left(\sigma_{res}^2 + \sigma_{p2}^2 \cdot \frac{c}{c_1}\right)^{1/2}(c_{12})^{1/2} = k\, c_{12}\, n_1 n_2,$$

from which it follows that

$$k = \frac{(\sigma_{p1}^2)^{1/2}(c_1)^{1/2} + \left(\sigma_{res}^2 + \sigma_{p2}^2 \cdot \dfrac{c}{c_1}\right)^{1/2}(c_{12})^{1/2}}{c} .$$

Having determined k, n_1 and n_2 can be solved.

Since the functions of the optimization problem defined by (1) and (8) are equivalent to the functions defined by (1) and (7), the derivation of the solution for this problem is analogous.

To apply the Cauchy-Schwarz inequality to the optimization problem defined by (1) and (9), both functions need to be rewritten. We start by dividing (9) by $n_1 n_2$ which gives $\frac{c_1}{n_2} + \frac{c_2}{n_1} = \frac{c}{n_1 n_2}$. By adding a constant to (1) and multiplying this constant by $\frac{c}{n_1 n_2}$, (1) can be expressed as

$$\frac{\sigma_{p1}^2}{n_1} + \frac{\sigma_{p2}^2}{n_2} + \frac{\sigma_{res}^2}{c} \cdot \frac{c}{n_1 n_2} = \frac{\sigma_{p1}^2}{n_1} + \frac{\sigma_{p2}^2}{n_2} + \frac{\sigma_{res}^2 c_1}{c n_2} + \frac{\sigma_{res}^2 c_2}{c n_1} =$$

$$\frac{\left(\sigma_{p1}^2 + \dfrac{\sigma_{res}^2 c_2}{c}\right)}{n_1} + \frac{\left(\sigma_{p2}^2 + \dfrac{\sigma_{res}^2 c_1}{c}\right)}{n_2} .$$

From (4) and (5) we obtain

$$\frac{\left(\sigma_{p1}^2 + \dfrac{\sigma_{res}^2 c_2}{c}\right)}{n_1} : c_1 n_1 = k^2, \text{ and} \tag{12}$$

$$\frac{\left(\sigma_{p2}^2 + \frac{\sigma_{res}^2 C_1}{C}\right)}{n_2} : C_2 n_2 = k^2. \tag{13}$$

Multiplying (12) by $(C_1)^{1/2}$ and (13) by $(C_2)^{1/2}$ results in

$$\left(\sigma_{p1}^2 + \frac{\sigma_{res}^2 C_2}{C}\right)^{1/2} (C_1)^{1/2} = k\, C_1\, n_1, \text{ and}$$

$$\left(\sigma_{p2}^2 + \frac{\sigma_{res}^2 C_1}{C}\right)^{1/2} (C_2)^{1/2} = k\, C_2\, n_2,$$

from which it follows that

$$k = \frac{\left(\sigma_{p1}^2 + \frac{\sigma_{res}^2 C_2}{C}\right)^{1/2} (C_1)^{1/2} + \left(\sigma_{p2}^2 + \frac{\sigma_{res}^2 C_1}{C}\right)^{1/2} (C_2)^{1/2}}{C}.$$

Having determined k, n_1 and n_2 can be solved.

The solution of the optimization problem defined by (1) and (2) also requires a restatement of the functions concerned. Dividing (2) by $n_1 n_2$ results in $\frac{C_1}{n_2} + \frac{C_2}{n_1} - \frac{C}{n_1 n_2} = -C_{12}$. By adding a constant to (1) and multiplying this constant by $-C_{12}$, (1) can be expressed as

$$\frac{\sigma_{p1}^2}{n_1} + \frac{\sigma_{p2}^2}{n_2} + \frac{\sigma_{res}^2}{n_1 n_2} + \frac{\sigma_{res}^2}{C} \cdot -C_{12} = \frac{\left(\sigma_{p1}^2 + \frac{\sigma_{res}^2 C_2}{C}\right)}{n_1} + \frac{\left(\sigma_{p2}^2 + \frac{\sigma_{res}^2 C_1}{C}\right)}{n_2}. \tag{14}$$

Adding $\frac{C_1 C_2}{C_{12}}$ to both sides of (2) results in

$$C_1\left(n_1 + \frac{C_2}{C_{12}}\right) + C_{12}\left(n_1 + \frac{C_2}{C_{12}}\right)n_2 = \left(C + \frac{C_1 C_2}{C_{12}}\right) \Rightarrow$$

$$C_{12} = -\frac{C_1}{n_2} + \frac{\left(C + \frac{C_1 C_2}{C_{12}}\right)}{\left(n_1 + \frac{C_2}{C_{12}}\right)n_2}.$$

By adding a constant and multiplying this constant by C_{12}, (14) can be expressed as

$$\frac{\left(\sigma_{p1}^2 + \frac{\sigma_{res}^2 C_2}{C}\right)}{n_1} + \frac{\left(\sigma_{p2}^2 + \frac{\sigma_{res}^2 C_1}{C}\right)}{n_2} + \frac{\left(\sigma_{p2}^2 + \frac{\sigma_{res}^2 C_1}{C}\right)}{C_1} \cdot C_{12} =$$

$$\frac{\left(\sigma_{p1}^2 + \frac{\sigma_{res}^2 C_2}{C}\right)}{n_1} + \frac{\left(C + \frac{C_1 C_2}{C_{12}}\right)\left(\frac{\sigma_{p2}^2 C + \sigma_{res}^2 C_1}{CC_1}\right)}{\left(n_1 + \frac{C_2}{C_{12}}\right)n_2}. \tag{15}$$

As cost function (2) can also be stated as

$$C_1 n_1 + C_{12}\left(n_1 + \frac{C_2}{C_{12}}\right)n_2 = C, \tag{16}$$

we obtain from (15) and (16),

$$\frac{\left(\sigma_{p1}^2 + \frac{\sigma_{res}^2 C_2}{C}\right)}{n_1} : C_1 n_1 = k^2, \text{ and} \tag{17}$$

$$\frac{\left(C + \frac{C_1 C_2}{C_{12}}\right)\left(\frac{\sigma_{p2}^2 C + \sigma_{res}^2 C_1}{CC_1}\right)}{\left(n_1 + \frac{C_2}{C_{12}}\right)n_2} : C_{12}\left(n_1 + \frac{C_2}{C_{12}}\right)n_2 = k^2. \tag{18}$$

Multiplying (17) by $(C_1)^{1/2}$ and (18) by $(C_{12})^{1/2}$ results in

$$\left(\sigma_{p1}^2 + \frac{\sigma_{res}^2 C_2}{C}\right)^{1/2} (C_1)^{1/2} = k\ C_1 n_1, \text{ and} \tag{19}$$

$$\left(C + \frac{C_1 C_2}{C_{12}}\right)^{1/2} \left(\frac{\sigma_{p2}^2 C + \sigma_{res}^2 C_1}{CC_1}\right)^{1/2} (C_{12})^{1/2} = k\ C_{12}\left(n_1 + \frac{C_2}{C_{12}}\right) n_2, \tag{20}$$

from which it follows that

$$k = \frac{\left(\sigma_{p1}^2 + \frac{\sigma_{res}^2 C_2}{C}\right)^{1/2} (C_1)^{1/2} + \left(C + \frac{C_1 C_2}{C_{12}}\right)^{1/2} \left(\frac{\sigma_{p2}^2 C + C_1 \sigma_{res}^2}{CC_1}\right)^{1/2} (C_{12})^{1/2}}{C}.$$

Having determined k, n_1 and n_2 can be solved.

Two-Facet Example

Using variance components $\hat{\sigma}_p^2 = 5.435$, $\hat{\sigma}_{p1}^2 = 3.421$, $\hat{\sigma}_{p2}^2 = 1.140$, and $\hat{\sigma}_{res}^2 = 11.850$, the objective-function is defined as:

minimize $$\frac{3.421}{n_1} + \frac{1.140}{n_2} + \frac{11.850}{n_1 n_2}.$$

Let C_1, C_2 and C_{12}, of cost functions (2), (7), (8), and (9) be the cost of a condition of Facet 1 (e.g., essay questions), the cost of a condition of Facet 2 (for example, raters), and the cost of one observation for all subjects in the sample (i.e., the answers of all students to one essay question rated by one rater), respectively. With a total budget of 3000 dollars, the following three cost functions, leading to three optimization problems, are considered: $40n_1 + 80n_1n_2 \leq 3000$, $30n_1 + 80n_2 \leq 3000$, and $40n_1 + 80n_2 + 80n_1n_2 \leq 3000$. The optimal integer solutions $\hat{n}_1$ and $\hat{n}_2$ for the three optimization problems are derived in two steps. First, the optimal continuous solutions are determined with the equations presented before. These solutions are $n_1^* = 8.8$ and $n_2^* = 3.8$ for the first problem, $n_1^* = 40.9$ and $n_2^* = 17$ for the second problem, and $n_1^* = 9.1$ and $n_2^* = 3.3$ for the third problem. Second, a branch-and-bound procedure such as described in Sanders et al. (1991) is used to obtain the optimal integer solutions. Table 1 contains the results for the three problems.

TABLE 1

Values of n_1, n_2, Variance Components, ρ^2 and C for Three Cost Functions.

n_1	c_1n_1	n_2	c_2n_2	n_1n_2	$c_{12}n_1n_2$	$\hat{\sigma}^2_p$	$\frac{\hat{\sigma}^2_{p1}}{n_1}$	$\frac{\hat{\sigma}^2_{p2}}{n_2}$	$\frac{\hat{\sigma}^2_{res}}{n_1n_2}$	ρ^2	C
1. c_1 = 40 dollars, c_2 = 0 dollars, and c_{12} = 80 dollars											
8.8	351	3.8	0	33.1	2649	5.435	.390	.302	.358	.838	3000
6	240	5	0	30	2400	5.435	.570	.228	.395	.820	2640
7	280	5	0	35	2800	5.435	.489	.228	.339	.837	3080
8*	320	4*	0	32	2560	5.435	.428	.285	.370	.834	2880
9	360	4	C	36	2880	5.435	.380	.285	.329	.845	3240
10	400	3	0	30	2400	5.435	.342	.380	.395	.830	2800
11	440	3	0	33	2640	5.435	.311	.380	.359	.838	3080
2. c_1 = 40 dollars, c_2 = 80 dollars, and c_{12} = 0 dollars											
40.9	1635	17	1364	697.5	0	5.435	.084	.067	.017	.970	3000
39*	1560	18*	1440	702	0	5.435	.088	.063	.017	.970	3000
40	1600	17	1360	680	0	5.435	.086	.067	.017	.970	2960
41*	1640	17*	1360	697	0	5.435	.083	.067	.017	.970	3000
3. c_1 = 40 dollars, c_2 = 80 dollars, and c_{12} = 80 dollars											
9.1	361	3.3	261	29.7	2375	5.435	.376	.350	.399	.829	3000
7*	280	4*	320	28	2240	5.435	.489	.285	.432	.819	2840
8	320	4	320	32	2560	5.435	.428	.285	.370	.834	3200
9	360	3	240	27	2160	5.435	.380	.380	.439	.819	2760
10	400	3	240	30	2400	5.435	.342	.380	.395	.830	3040

*Optimal integer solutions

Conclusions and Discussion

With respect to cost functions, the influence of the cost factors on the optimal solution should be noted. From the solutions in Table 1 it can be seen that the cost functions discussed here are dominated by cost factor c_{12} = 80 dollars. The solutions for the first problem with cost function (7) and the third problem with cost function (2) are therefore very close, while they are quite different from the solutions for the second problem with cost function (9). These results, however, also imply that an optimization problem with a cost function including a dominant cost factor will give good starting solutions for a branch-and-bound procedure for other optimization problems with cost functions including this dominant cost factor.

In survey sampling, (7) or (8) and (9) are the cost functions employed in connection with the allocation of optimal resources for designs known as two-stage and two-phase sampling designs. Applications of the Cauchy-Schwarz inequality to more complex designs, for example, three-stage sampling designs (e.g., Snedecor & Cochran 1976, p. 533) have been presented. As this note showed the formal similarity between optimization problems in survey sampling and generalizability theory, these applications signify that solutions for more complex designs employed in generalizability theory could also be derived.

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