Report Nr 04

Minimizing the number of observations: a generalization of the Spearman-Brown formula

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MINIMIZING THE NUMBER OF OBSERVATIONS:
A GENERALIZATION OF THE SPEARMAN-BROWN FORMULA

by

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Report Nr 04 (To appear in Psychometrika)

Project 'Optimal Item Selection'

CITO

Arnhem, the Netherlands (1988)

General Introduction

The purpose of Project 'Optimal Item Selection' is to solve a number of issues in automated test design, making extensive use of optimization techniques. To this end, there has been close cooperation between the project and, among others, the department of Operations Research at Twente University. In each report, one or several theoretical issues are raised and an attempt is made to solve them. Furthermore, each report is accompanied by one or more computer programs, which are the implementations of the methods that have been investigated. In due time, requests for these programs can be sent to the project director.

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project director.

Abstract

A new method for determining the minimum number of observations per subject needed to achieve a specific generalizability coefficient is presented. This method, which consists of a branch-and-bound algorithm, allows for the employment of constraints specified by the investigator.

Introduction

In generalizability theory (Cronbach et al., 1972) a distinction is made between a generalizability (G) study and a decision (D) study. In a G-study, estimates of variance components are obtained which can be used by the investigator in a D-study. One of the major decisions an investigator has to make is how many observations per subject or another object of measurement are necessary in order to control the principal sources of random sampling error or to achieve a given generalizability coefficient (cf. Cardinet and Allal, 1983, p. 42).

For one-facet designs the minimum number of observations per subject can be determined as follows. The coefficient of reliability for the one-facet random-model crossed design, ρ^2, may be expressed as:

$$\rho^2 = \frac{\sigma^2_p}{\sigma^2_p + \frac{\sigma^2_{res}}{n_1}} \tag{1}$$

where σ^2_p is the variance component for persons, σ^2_{res} is the variance component for the p x facet 1 interaction plus the error, and n_1 is the number of observations, i.e., conditions of facet 1, in the D-study. Rewriting (1) and letting ρ^2 be a specific reliability coefficient, the minimum number of observations per subject is equal to:

$$n_1 = \frac{\rho^2\sigma^2_{res}}{\sigma^2_p - \rho^2\sigma^2_p} \tag{2}$$

Equations (1) and (2) both exemplify the Spearman-Brown Prophecy

Formula from classical test theory: increase/decrease of the number of observations, e.g., items, results in an increase/decrease of the reliability coefficient. This correspondence between number of observations and reliability, however, does not extend to designs with more than one facet. Increasing the number of conditions of a facet with a large error variance component, for example, will have a greater impact on the generalizability coefficient than increasing the number of conditions of a facet with a small error variance component. With multi-facet designs it is therefore possible to increase the generalizability coefficient while decreasing the number of observations. The multi-dimensional nature of error variance is the reason for this paradoxical result, which is inconsistent with assumptions in classical test theory but not with assumptions in generalizability theory (cf. Brennan, 1983, p. 67).

Because of the multi-dimensional nature of error variance in generalizability theory, the determination of the minimum number of observations is much more complex for multi-facet designs than for one-facet designs. Woodward and Joe (1973) presented a method for solving this problem, which they illustrated for the two-facet random-model crossed design. The generalizability coefficient for this design may be expressed as:

$$\rho^2 = \frac{\sigma^2_p}{\sigma^2_p + \frac{\sigma^2_{p1}}{n_1} + \frac{\sigma^2_{p2}}{n_2} + \frac{\sigma^2_{res}}{n_1 n_2}} \qquad (3)$$

where σ^2_p is the variance component for persons, σ^2_{p1} is the variance component for the person by facet 1 interaction, σ^2_{p2} is the variance component for the person by facet 2 interaction, σ^2_{res} is the variance component for the p x facet 1 x facet 2 interaction plus error, n_1 and n_2 are the number of conditions of facet 1 and facet 2 in the D-study. Denoting the total number of observations for this design by $L = n_1 n_2$, the product of the number of conditions, (3) can be written as:

$$L = \frac{\rho^2 (\sigma^2_{p1} n_2 + \sigma^2_{res})}{\sigma^2_p - \rho^2 \sigma^2_p - \rho^2 \sigma^2_{p2}/n_2} \qquad (4)$$

Woodward and Joe's method consists of taking the derivative of this function with respect to n_2 and setting the result equal to zero, thus obtaining the following quadratic equation:

$$0 = (\sigma^2_p \sigma^2_{p1}) \frac{1 - \rho^2}{\rho^2} n_2^2 - (2\sigma^2_{p1} \sigma^2_{p1}) n_2 - \sigma^2_{res} \sigma^2_{p1} \qquad (5)$$

The positive root of the equation is taken as the desired solution and L is found by substituting the positive n_2 into (4).

The results of their method for a two-facet random-model crossed design with $\hat{\sigma}^2_p = 5.435$, $\hat{\sigma}^2_{p1} = 3.421$, $\hat{\sigma}^2_{p2} = 1.140$ and $\hat{\sigma}^2_{res} = 11.850$ are presented in Table 1.

Table 1: Values of minimum L and optimal values of n_1 and n_2 for various values of ρ^2

ρ^2	L	n_1	n_2
.97	685.9	45.4	15.1
.96	402.1	34.7	11.6
.95	267.1	28.7	9.4
.94	192.0	24.0	8.0
.93	145.4	20.9	6.9
.92	114.5	18.5	6.2
.91	92.8	16.7	5.6
.90	77.0	15.2	5.1
.89	65.1	13.9	4.6
.88	55.8	12.9	4.3
.87	48.4	12.1	4.0
.86	42.5	11.3	3.8
.85	37.6	10.6	3.5
.84	33.5	10.0	3.3
.83	30.1	9.5	3.2
.82	27.0	9.0	3.0
.81	24.7	8.6	2.8
.80	22.5	8.2	2.7
.79	20.6	7.8	2.6
.78	18.9	7.5	2.5
.77	17.5	7.2	2.4
.76	16.1	6.9	2.3
.75	14.9	6.7	2.2
.74	13.9	6.4	2.2
.73	13.0	6.2	2.1
.72	12.1	6.0	2.0
.71	11.3	5.8	1.9
.70	10.6	5.6	1.8

The method proposed by Woodward and Joe, however, has a number of shortcomings. Firstly, the method requires derivations and calculations which can become quite complex for designs with more than two facets, nested designs and designs based on other models. Secondly, the values obtained for the number of conditions of different facets and L are usually non-integer. Woodward and Joe's solution for this problem consists of rounding off the values obtained for n_1 and n_2 to the nearest whole numbers. By rounding off, however, their conclusion that the positive root of the quadratic equation will always give the smallest L no longer applies. Consequently, the values for n_1 and n_2 resulting from their method will not always be the optimal values. For instance, specifying a value of .79 for the generalizability coefficient and rounding off the values of n_1 and n_2 to 8 and 3 will result in L being equal to 24. The specified value of .79, however, could also have been obtained with 7 conditions for facet 1 and 3 conditions for facet 2, resulting in L = 21 observations. (The generalizability coefficients for these and other values for the number of conditions referred to in this article are presented in Table 2.)

Table 2: Various values of n_1, n_2, L with resulting variance components and ρ^2

n_1	n_2	L	$\hat{\sigma}^2_p$	$\frac{\hat{\sigma}^2_{p1}}{n_1}$	$\frac{\hat{\sigma}^2_{p2}}{n_2}$	$\frac{\hat{\sigma}^2_{res}}{n_1 n_2}$	ρ^2
6	4	24	5.4	.57017	.285	.49375	.80166
6	6	36	5.4	.57017	.190	.32917	.83303
7	3	21	5.4	.48871	.380	.56429	.79135
7	4	28	5.4	.48871	.285	.42321	.81952
8	3	24	5.4	.42763	.380	.49375	.80681
9	3	27	5.4	.38011	.380	.43889	.81926
10	3	30	5.4	.34210	.380	.39500	.82951
11	3	33	5.4	.31100	.380	.35909	.83808
12	2	24	5.4	.28508	.570	.49375	.80117
24	2	48	5.4	.14254	.570	.24688	.84996
36	2	72	5.4	.09503	.570	.16458	.86757

A difference of one condition for one facet can make a substantial difference in resources. For instance, with a two-facet crossed design the difference of one condition could mean that one rater less is needed to correct the answers of a hundred students to ten questions. That the generalizability coefficients of the two designs hardly differ is due to the insensitivity of higher values of the coefficient to even major changes in the design. It is clear that for designs with more than two facets, the difference of one condition for one facet can make an even more dramatic difference in resources. A third and more serious shortcoming of the method, however, is that it implies a very restrictive constraint on the values for n_1 and n_2. Table 1 shows that the values for n_1 and n_2 are determined by the ratio of the variance components σ^2_{p1} and σ^2_{p2}, i.e., the ratio $\hat{\sigma}^2_{p1}/\hat{\sigma}^2_{p2}$=3.421/1.140. Therefore more than three times as many conditions are allocated to n_1 than to n_2.

The method proposed here is different from Woodward and Joe's method in a practical as well as theoretical respect. The new method consists of an algorithm based on the concept of enumeration, using a branch-and-bound algorithm, the principles of which are well known in integer programming (e.g., Salkin, 1975). The method is presented in three parts. First, the structure of a branch-and-bound algorithm is described. Next, an algorithm for a two-facet random-model crossed design is presented. Finally, the algorithm for the two-facet design is generalized for multi-facet designs.

Branch-and-bound algorithm

The term branch-and-bound algorithm does not refer to one specific algorithm but to a class of algorithms. Papadimitriou and Steiglitz (1985, p.433) describe the branch-and-bound approach as the construction of a proof that a solution is optimal based on successive partitioning of the solution space. The parts branch and bound refer to rules which reduce the amount of search to be conducted for the optimal solution. A branch-and-bound algorithm is usually represented by a tree composed of branches and nodes, with the nodes organized in levels. In the tree as it is organized for the problem considered here, level ℓ, $\ell=1,2,\ldots,t$, corresponds with variable n_ℓ. Each node at level ℓ represents a partial solution in which variables $n_1,n_2,\ldots,n_\ell$ have fixed values, say $n_i=\hat{n}_i$, $i=1,2,\ldots,\ell$, whereas the remaining $t-\ell$ variables are said to be free, to indicate that their values still need to be determined in the further course of the search-process. The root of the full search-tree consists of a single node at level 0 in which all variables are free. The nodes at the highest level (level t) correspond with complete solutions of the problem and may therefore all be regarded as candidates for the optimal solution. A node at level $\ell \leq t-1$, with partial solution $\hat{n}_1,\hat{n}_2,\ldots,\hat{n}_\ell$, is connected by branches with all nodes at level $\ell+1$ in which the associated partial solution is different only in the fixed value of variable $n_{\ell+1}$, which was free before. Starting from the root of the tree, a complete solution can gradually be developed by passing through individual nodes, one at each level, until level t is reached. In this way any specific node at level t is reachable along a unique path from the root. On the other hand, from any node at an intermediate level $\ell < t$ several nodes at level t may be attained. In general the tree and thus the number of complete solutions will increase rapidly as the number of variables increases. What is needed therefore are additional rules that will allow a significant reduction of the full search-tree by cutting off those parts which are irrelevant regarding an optimal solution.

Branch and bound for the two-facet design problem

The problem for the two-facet design can be stated in terms of mathematical optimization as:

minimize	$L = n_1 n_2$	objective-function	(6)
subject to	$\rho^2(n_1,n_2) \geq g$	threshold constraint	(7)
	$n_1 \geq n_2 \geq 1$	monotonicity constraints	(8)
	n_1 and n_2 integer	integer constraint	(9)

In the minimization statement (6) of this optimization problem, L refers to the value of the objective-function which results when different numbers of conditions, n_1 and n_2, for facet 1 and 2 are used. Note that L can also be interpreted as minimizing the cost of L observations. If c_1 and c_2 are the costs associated with n_1 and n_2 respectively, then $L = c_1 c_2 n_1 n_2$ corresponds with the total cost of L observations. However, since $c_1 c_2$ is a constant, it will not influence the result of the minimization procedure and may therefore be omitted.

In the threshold constraint (7), $\rho^2(n_1,n_2)$ stands for the generalizability coefficient of a two-facet random-model crossed design and g for the lowest acceptable value of a generalizability coefficient. The function $\rho^2(n_1,n_2)$ is strictly increasing with respect to both variables.

The monotonicity constraint (8) $n_1 \geq n_2$ employed here is but one of many linear inequality constraints that could be employed. Note that an optimal solution for the two-facet design problem can also be obtained without this constraint. However, an algorithm employing this constraint will exclude an irrelevant part of the decision-space and consequently reduce the number of branchings in the branch-and-bound process described hereafter.

The integer constraint (9) states that feasible values for n_1 and n_2 have to be integer values.

After the problem has been formulated as an optimization problem, bounding rules are constructed which effectively reduce the search-process. A distinction can be made between feasibility bounds, i.e., bounds on the values that n_1 and n_2 can assume without violating the constraints, and optimality bounds, bounds that use a comparison of objective-function values to ascertain whether a given partial solution can lead to an optimal solution. Both types of bounds will be derived in this paper.

To fix the number of relevant branches emanating from the root of the search-tree, a lower bound ℓb_1 and an upper bound ub_1 on the values that n_1 can assume in an optimal solution can be derived as follows. Regarding the threshold constraint, it can easily be seen that $\rho^2(n_1,n_2)$ is strictly increasing both in n_1 and n_2 so that if $n_1 \geq n_1^*$ and $n_2 \geq n_2^*$, then $\rho^2(n_1,n_2) \geq \rho^2(n_1^*,n_2^*)$ while strict inequality holds whenever $n_1 > n_1^*$ or $n_2 > n_2^*$. Hence for a given value $\hat{n}_1$ of n_1, there either exists a least integer value $\hat{n}_2 \leq \hat{n}_1$ so that the threshold constraint is satisfied, or for all $n_2 \leq \hat{n}_1$ this constraint is violated. This observation implies the existence of a value n^-

such that $\rho^2(n_1=n^-,n_2=n^-) \geq g$, whereas for all $n_1 < n^-$ no value $n_2 \leq n_1$ exists that satisfies the threshold constraint. This means that n^- should be taken as the lower bound for facet 1. For instance, with (3) it can be calculated that for Woodward and Joe's example $n_1=6$ is the lowest integer value such that $\rho^2(6,6) \geq .80$, while for $n_1 \leq 5$ and for each value $n_2 \leq n_1$ one finds $\rho^2(n_1,n_2) < .80$. Consequently, in this case $\ell b_1=n^-=6$. Note that without the integer constraint $n_1=n_2=n^-=5.08$.

Let $L^-=n_1^-n_2^-$ be the value of the objective-function associated with the solution $n_1=n_2=n^-$. Then $ub_1=L^-$ is an obvious upper bound on the values of n_1, because any value $n_1 > ub_1$ produces a solution that has a value $n_1n_2 > ub_1=L^-$ and is therefore irrelevant for optimality. While initially ub_1 may thus be intractably high, its value can be adapted throughout the search-process whenever a new and better solution is found. The current best solution is called the incumbent. So $n_1=n_2=n^-$ as defined above is the initial incumbent. As soon as a complete solution (n_1^*,n_2^*) with value $L^*=n_1^*n_2^* < L^-$ is derived, this becomes the new incumbent and $ub_1=L^*$ replaces the initial upper bound. This process with repeated successive adaptations of both the incumbent and ub_1 is continued until the whole search-tree has been explored. The then operative incumbent is designated as the optimal solution.

In the example, $ub_1=n^-n^-=36$ becomes the first upper bound for n_1. With $\ell b_1=6$ and $ub_1=36$ the total number of nodes at level 1 becomes $ub_1-\ell b_1+1=31$. Any such node represents a partial solution with a fixed value for n_1 and n_2 free (see Figure 1). Now, for any node at level 1 corresponding with a fixed value $\hat{n}_1$ for n_1 there is a least integer value $\hat{n}_2$ for n_2 such that $\rho^2(\hat{n}_1,\hat{n}_2) \geq g$. Since this value minimizes $L=\hat{n}_1 n_2$ subject to the constraints $\rho^2(\hat{n}_1,n_2) \geq g$ and $\hat{n}_1 > n_2 \geq 1$, it suffices to consider a tree in which each node at level 1 corresponds with precisely one node at level 2. For instance, with (3) the unique feasible value for n_2 that minimizes $\hat{n}_1 n_2$ given $\hat{n}_1=8$ is found to be $\hat{n}_2=3$.

By convention, the branch associated with the lower bound is further referred to as 'left-most' branch and the one associated with the upper bound as 'right-most' branch. All branches between level 0 (the root) and level 1 are further ordered according to increasing values of n_1. The search-tree for the two-facet problem is now traversed as follows. Starting at the root of the tree, the left-most branch to level 1 is considered first, immediately followed by the unique branch that connects the node with partial solution $\hat{n}_1=\ell b_1$ to the corresponding node at level 2 with complete solution $n_1=\hat{n}_1=\ell b_1, n_2=\hat{n}_2$. Next the process returns to the root, which for the two-facet problem is the most recent node from which not all branches to the next level have been considered yet. This is represented by saying that the root is not fathomed. Note that each node at level 1 is fathomed as soon as the search-process has passed through it once, due to the fact that there is only one branch leading to level 2.

The two nodes regarded next are the one at level 1 with partial solution $\hat{n}_1=\ell b_1+1$, and its corresponding node at level 2. The full search-process as it is conducted for the example with $g \geq .80$ is shown in Figure 1. In this figure the nodes are numbered in their order of appearance.

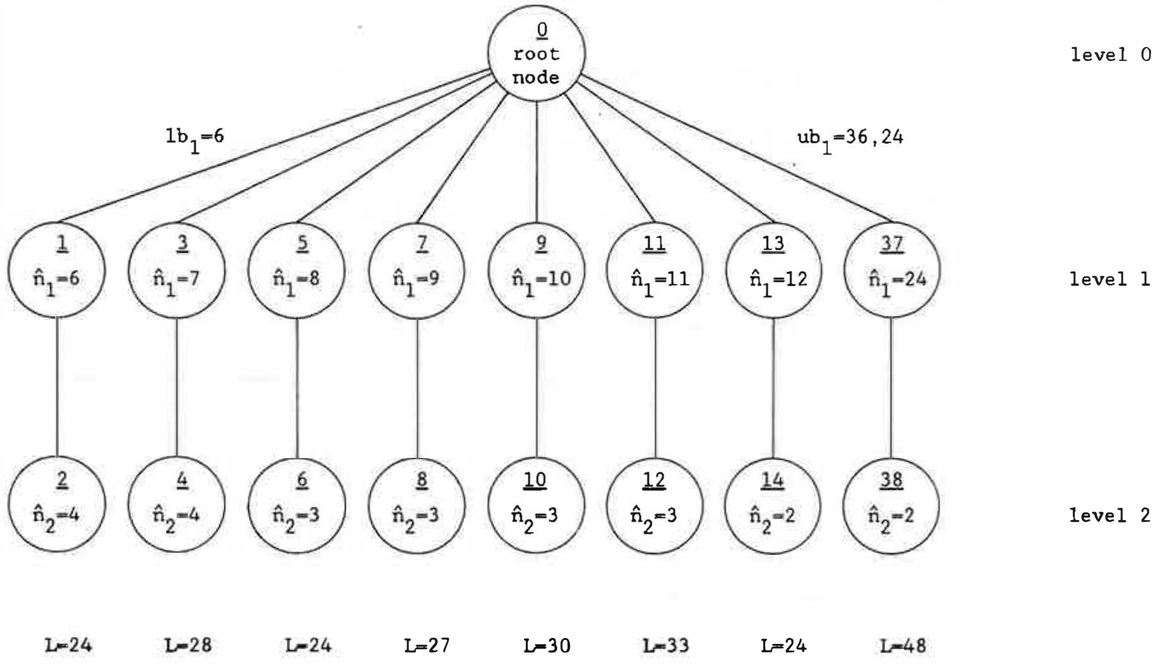

Figure 1: Search-tree for the two-facet example

The initial incumbent, whose value is 36, is replaced in node 2 by $(\hat{n}_1,\hat{n}_2)=(6,4)$, giving the new incumbent the value 24. Further exploration of the search-tree renders two more solutions, producing the same value for the objective-function as node 2: $(\hat{n}_1,\hat{n}_2)=(8,3)$ in node 6 and $(\hat{n}_1,\hat{n}_2)=(12,2)$ in node 14. The search-process ends in node 38 with solution $(\hat{n}_1,\hat{n}_2)=(24,2)$, yielding the objective-function value 48 which is higher than the incumbent.

A closer inspection of the search-tree shown in Figure 1 reveals that a considerable reduction can be obtained regarding the size of the tree by eliminating irrelevant nodes and branches. For example, all nodes from 15 up to and including 38 are superfluous, as are all corresponding branches between level 1 and 2, since they give rise to solutions in which n_2 remains constant at a value $\hat{n}_2=2$, implying objective-function values $\geq 13 \times 2=26 >$

24. This observation is clearly related to the fact that from each node at level 1 there is a unique branch passing to level 2. This allows for the formulation of the following optimality bound. Let the values of n_1 at level 1, further referred to as $\hat{n}_{1,i}$, $i=1,2,\ldots,s=ub_1-\ell b_1+1$, be such that $\hat{n}_{1,1}=\ell b_1$, $\hat{n}_{1,2}=\ell b_1+1,\ldots,\ \hat{n}_{1,i}=\ell b_1+i-1,\ldots,\ \hat{n}_{1,s}=ub_1$. Accordingly, for $i=1,2,\ldots,s$, $\hat{n}_{2,i}$ is the value of n_2 that is uniquely associated with the partial solution $\hat{n}_{1,i}$, in a way described before. Then as long as $\hat{n}_{2,i}$ remains unaltered while $\hat{n}_{1,i}$ steadily increases by values of i, there will certainly be no improvement in the value of the objective-function. The objective-function value can only be improved when $\hat{n}_{2,i} > \hat{n}_{2,i+1}$ for some i. So let $i_0,i_1,\ldots,i_q$ be such that $i_0=1$ and $\hat{n}_{2,i_0}=\hat{n}_{2,1}=\ldots=\hat{n}_{2,i_1-1} > \hat{n}_{2,i_1}=\hat{n}_{2,i_1+1}=\ldots=\hat{n}_{2,i_2-1} > \hat{n}_{2,i_2} =\ldots= \hat{n}_{2,i_q-1} > \hat{n}_{2,i_q}=\hat{n}_{2,i_q+1}=\ldots=\hat{n}_{2,s}$. For our example it can be inferred that $4=\hat{n}_{2,1} = \hat{n}_{2,2} > 3=\hat{n}_{2,3}=\hat{n}_{2,4}=\hat{n}_{2,5}= \hat{n}_{2,6} > 2=\hat{n}_{2,7} =\ldots=\hat{n}_{2,19}$. The lowest index i for which $\hat{n}_{2,i}=3$ is therefore equal to $i_1=3$ and the lowest index for which $\hat{n}_{2,i}=2=\hat{n}_{2,s}$ is equal to $i_2=7$, implying q=2. These results give rise to a search-tree in which only the nodes at level 1 corresponding with values $\hat{n}_{1,i_0},\hat{n}_{1,i_1},\ldots,\ \hat{n}_{1,i_q}$ and the associated nodes at level 2 are considered. The values of i_0, $i_1,\ldots,\ i_q$ can be derived as follows. After $\hat{n}_{1,1}=\ell b_1$ has been established, the associated value of $\hat{n}_{2,1}$ as the least integer value such that $\rho^2(\hat{n}_{1,1},\hat{n}_{2,1}) \geq g$ can be calculated with (3). Let $ub_2=\hat{n}_{2,1}$. Now regarding a value ub_2-1 for n_2, i_1 is chosen as the lowest index such that $\hat{n}_{1,i_1} \leq ub_1 = \hat{n}^2_{1,1}$ and $\rho^2(\hat{n}_{1,i_1},ub_2-1) \geq g$. By generalizing this rule, the value of i_j becomes the lowest index such that $\hat{n}_{1,i_j} \leq ub_1$ and $\rho^2(\hat{n}_{1,i_j},ub_2-j) \geq g$,

$j=0,1,\ldots,ub_2-q(q\geq 1)$. Using this optimalitity bound in our example results in a reduced search-tree which is presented in Figure 2, where the numbering of the nodes refers to the nodes in Figure 1.

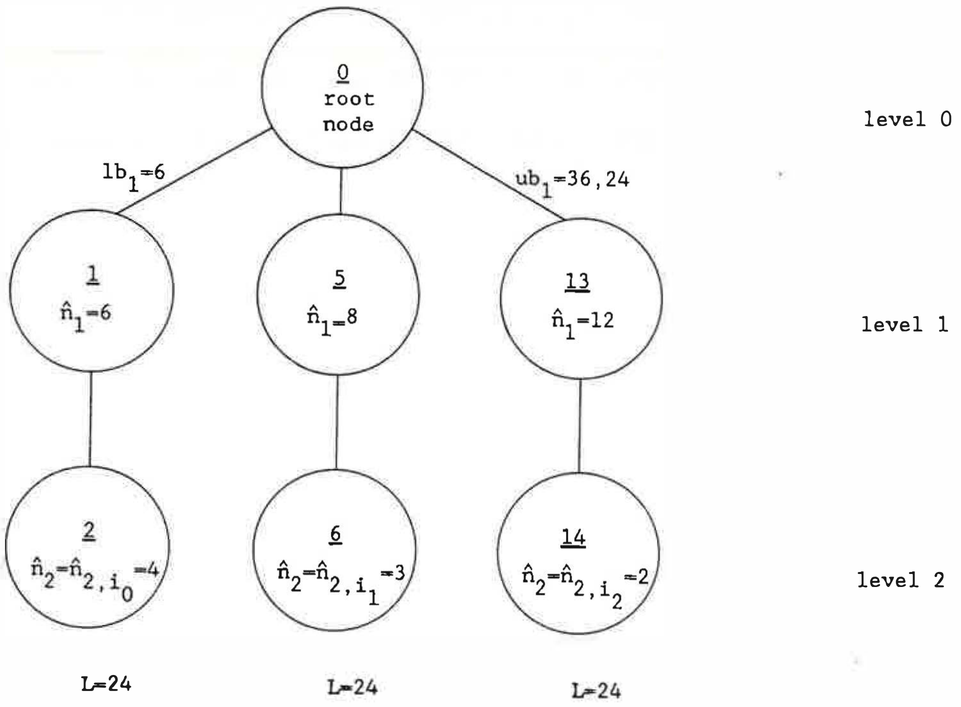

Figure 2: Reduced search-tree for the example $(i_0=1,i_1=3,i_2=7)$

At the end of the search-process there appear to be three candidates for an optimal solution: $(\hat{n}_1,\hat{n}_2)=(6,4)$, $(\hat{n}_1,\hat{n}_2)=(8,3)$ and $(\hat{n}_1,\hat{n}_2)=(12,2)$. Solution $(\hat{n}_1,\hat{n}_2)=(8,3)$ could be considered the 'most optimal' solution because it results in a higher generalizability coefficient than the other two solutions. However, considerations other than obtaining a specific generalizability coefficient can and often will play a role when constructing a measurement instrument. If in this example facet 1 had been items and facet 2 raters, there could have been considerable differences in the costs per condition of these two facets. Raters probably being more expensive than items, an investigator could for economic reasons prefer to increase the number of items rather than to increase the number of raters. This should be indicated by employing an economic constraint such as $n_1 \geq 5n_2$ instead of constraint $n_1 \geq n_2$ which is a psychometric constraint. Using the above specification $g \geq .80$ and employing the economic constraint mentioned, the optimal solution appears to be $(\hat{n}_1,\hat{n}_2)=(12,2)$.

Branch and bound for the multi-facet design problem

The branch-and-bound method developed for the two-facet problem can be generalized for the multi-facet problem as follows:

minimize	$L(n_1,n_2,\ldots,n_t)=\prod_{i=1}^{t} n_i = n_1 n_2 \ldots n_t$	objective function
subject to	$\rho^2(n_1, n_2, \ldots, n_t) \geq g$	threshold constraint
	$n_1 \geq n_2 \geq \ldots \geq n_t \geq 1$	monotonicity constraints
	$n_1, n_2, \ldots, n_t$ integer	integer constraint

The level-wise organization of a search-tree for a branch-and-bound algorithm to solve this multi-facet problem is analogous to the organization of the search-tree for the two-facet problem in the foregoing section. Let k be an arbitrary node at level ℓ, with associated partial solution $\hat{n}_1,\hat{n}_2,\ldots,\hat{n}_\ell$. The branches passing from this node to level $\ell+1$ represent feasible values for $n_{\ell+1}$, given the partial solution. The most important question to be answered next is how to obtain both a lower bound $\ell b(\ell,k)$ and an upper bound $ub(\ell,k)$ on relevant values for $n_{\ell+1}$. Writing $\rho^2_\ell(n_{\ell+1},n_{\ell+2},\ldots,n_t)=\rho^2(\hat{n}_1,\hat{n}_2,\ldots,\hat{n}_\ell,n_{\ell+1},n_{\ell+2},\ldots,n_t)$ let n^- be the least integer value such that $\rho^2_\ell(n^-,n^-,\ldots,n^-) \geq g$. Then, using the fact that ρ^2_ℓ is strictly increasing in all its arguments, it can be easily verified that for each $n_{\ell+1} < n^-$ there exists no feasible completion of the partial solution $\hat{n}_1,\hat{n}_2,\ldots,\hat{n}_\ell$. For if such a completion existed, with $n_i=\hat{n}_i, i=\ell+1,\ell+2,\ldots,t$ and $n^- > \hat{n}_{\ell+1} \geq \hat{n}_{\ell+2} \geq \ldots \geq \hat{n}_t \geq 1$, then a 'constant' completion with $n_i=\hat{n}_{\ell+1}, i=\ell+1,\ell+2,\ldots,t$ would also be feasible, especially since $\rho_\ell^2(\hat{n}_{\ell+1},\hat{n}_{\ell+1},\ldots,\hat{n}_{\ell+1}) \geq \rho_\ell^2(\hat{n}_{\ell+1},\hat{n}_{\ell+2},\ldots,\hat{n}_t) \geq g$. This obviously contradicts the choice of n^- as a least constant integer value yielding a feasible completion. Consequently, $\ell b(\ell,k)=n^-$ can be regarded as a proper lower bound. In conformity with the two-facet design problem, an initial upper bound $ub(\ell,k)$ is obtained by putting $n_{\ell+1}=n_{\ell+2}=\ldots=n_t=n^-$ yielding $ub(\ell,k)=\hat{n}_1\hat{n}_2\hat{n}_3\ldots\hat{n}_\ell n^-\ldots n^- n^-$ ($t-\ell$ times n^-). This upper bound is adapted each time a lower objective-function value is obtained.

To realize a further reduction in the size of the tree consider a node k at level t-2. This node may be regarded as the root of a sub-tree in which the nodes at level t-1 correspond with values satisfying $\ell b(t2,k) \leq n_t \leq ub(t-2,k)$. In fact, a situation occurs that does not differ essentially from the two-facet case in the previous section. Using arguments similar to those used in that case, it is possible to determine for each node at level t-1 belonging to the sub-tree, a unique node at level t that represents an optimal completion of the associated partial solution. Moreover, the number of relevant nodes in the sub-tree at level t-1 and thus the number of branches between level t-2 and level t-1 may again be considerably reduced using a type of optimality bound similar to that used for the two-facet case.

The foregoing considerations give rise to straightforward generalizations of the principles underlying the branch-and-bound algorithm for the two-facet problem. They actually fix the structure of the search-tree for the multi-facet problem. What remains to be elucidated is how to perform subsequent steps such that in the end the full tree is most efficiently traversed. For that purpose a so-called depth-first strategy appears most plausible. This principle can be simply interpreted as follows. Let node k at level $\ell < t$ be the node that the branching process has just reached. Then, before the process returns to the node at level ℓ-1 immediately preceding k, the full sub-tree rooted in k is explored. If, in addition, the order in which the branches are covered is agreed upon, for example from 'left' (the branch associated with $\ell b(\ell,k)$) to 'right' ($ub(\ell,k)$), the course of the search-process is completely fixed. As in the two-facet case an incumbent, or current best solution, is retained and adapted whenever relevant. To initialize a constant solution $n_i = n^-$, $i=1,2,\ldots,t$, with n^- the least integer value such that $\rho^2(n^-, n^-, \ldots, n^-) \geq g$, suggests itself, in which n^- corresponds with the lower bound on values of n_1.

The branch-and-bound method for the multi-facet problem is illustrated by the three-facet example described in Cronbach (1972, p. 171 ff.), where the aphasic symptoms of 30 patients are rated by 4 raters (facet 2) on 6 graphic subtests (facet 3) using the same 10 objects (facet 1).

The generalizability coefficient for the three-facet random-model crossed design may be expressed as:

$$\rho^2 = \frac{\sigma^2_p}{\sigma^2_p + \frac{\sigma^2_{p1}}{n_1} + \frac{\sigma^2_{p2}}{n_2} + \frac{\sigma^2_{p3}}{n_3} + \frac{\sigma^2_{p12}}{n_1 n_2} + \frac{\sigma^2_{p13}}{n_1 n_3} + \frac{\sigma^2_{p23}}{n_2 n_3} + \frac{\sigma^2_{res}}{n_1 n_2 n_3}} \qquad (10)$$

where σ^2_p is the variance component for patients and the other components are the interaction components divided by the number of conditions being used in the D-study.

The problem for the three-facet design is stated as:

minimize	$L = n_1 n_2 n_3$	objective-function	(11)
subject to	$\rho^2(n_1, n_2, n_3) \geq g$	threshold constraint	(12)
	$n_3 \geq n_1 \geq n_2 \geq 1$	monotonicity constraints	(13)
	n_1, n_2 and n_3 integer	integer constraint	(14)

Note that the remarks made on the monotonicity contraints employed in the two-facet example (p. 8) also apply for the monotonicity constraints employed in the three-facet example.

The results for this example with $\hat{\sigma}^2_p = 5.3$, $\hat{\sigma}^2_{p1} = .41$, $\hat{\sigma}^2_{p2} = .15$, $\hat{\sigma}^2_{p3} = 1.71$, $\hat{\sigma}^2_{p12} = .15$, $\hat{\sigma}^2_{p13} = 2.01$, $\hat{\sigma}^2_{p23} = .07$ and $\sigma^2_{res} = .99$ and $g \geq .90$ are presented in Table 3.

Table 3: Three different constraints with resulting values for n_1, n_2, n_3, L, variance components and ρ^2

n_1	n_2	n_3	L	$\hat{\sigma}^2_p$	$\frac{\hat{\sigma}^2_{p1}}{n_1}$	$\frac{\hat{\sigma}^2_{p2}}{n_2}$	$\frac{\hat{\sigma}^2_{p3}}{n_3}$	$\frac{\hat{\sigma}^2_{p12}}{n_1 n_2}$	$\frac{\hat{\sigma}^2_{p13}}{n_1 n_3}$	$\frac{\hat{\sigma}^2_{p23}}{n_2 n_3}$	$\frac{\hat{\sigma}^2_{p123}}{n_1 n_2 n_3}$	ρ^2
$n_3 \geq n_1 \geq n_2 \geq 1$												
5	3	5	75	5.3	.08200	.05000	.34200	.01000	.08040	.00467	.01320	.90101
4	3	6	72	5.3	.10250	.05000	.28500	.01250	.08375	.00389	.01375	.90577
5	2	6	60	5.3	.08200	.07500	.28500	.01500	.06700	.00583	.01650	.90655
4	2	7	56	5.3	.10250	.07500	.24429	.01875	.07179	.00500	.01768	.90831
6	1	7	42	5.3	.06800	.15000	.24429	.02500	.04786	.01000	.02357	.90301
5	1	8	40	5.3	.08200	.15000	.21375	.03000	.05025	.00875	.02475	.90451
4	1	9	36	5.3	.10250	.15000	.19000	.03750	.05583	.00778	.02750	.90273
3	1	12	36	5.3	.13666	.15000	.14250	.05000	.05583	.00583	.02750	.90315
$n_1 \geq n_3 \geq n_2 \geq 1$												
5	3	5	75	5.3	.08200	.05000	.34200	.01000	.08040	.00467	.01320	.90101
6	2	5	60	5.3	.06800	.07500	.34200	.01250	.06700	.00700	.01650	.90014
7	1	7	49	5.3	.05857	.15000	.24429	.02143	.04102	.01000	.02020	.90668
8	1	6	48	5.3	.05125	.15000	.28500	.01875	.04187	.01167	.02062	.90149
$n_1 \geq n_2 \geq n_3 \geq 1$												
5	5	5	125	5.3	.08200	.03000	.34200	.00600	.08040	.00280	.00792	.90581

To illustrate our method, three different constraints were employed. The solutions in Table 3 show that the psychometric constraint $n_3 \geq n_1 \geq n_2 \geq 1$ results in L = 36 observations. This number of observations corresponds with solutions $(\hat{n}_1,\hat{n}_2,\hat{n}_3)=(4,1,9)$ and $(\hat{n}_1,\hat{n}_2,\hat{n}_3)=(3,1,12)$. However, these solutions are also the most expensive solutions since they involve the construction of new tests. An investigator who will therefore probably want to use the six tests that are already available should add constraint $n_3 \leq 6$ to this constraint, obtaining as an optimal solution $(\hat{n}_1,\hat{n}_2,\hat{n}_3)=(5,2,6)$ with L = 60. A more economical instrument would be one composed of as many objects and as few raters as possible and using no more tests than are available. Employing the corresponding economic constraints $n_1 \geq n_3 \geq n_2 \geq 1$ and $n_3 \leq 6$ would result in the optimal solution $(\hat{n}_1,\hat{n}_2,\hat{n}_3)=(8,1,6)$ with L = 48. Employing another economic constraint $n_1 \geq n_2 \geq n_3 \geq 1$ would result in the optimal solution $(\hat{n}_1,\hat{n}_2,\hat{n}_3)=(5,5,5)$ with L = 125 observations.

Conclusions and discussion

The method proposed in this article enables an investigator to specify an acceptable threshold for generalizability coefficients. The employment of the threshold constraint together with the integer constraint necessarily results in values for the objective-function and values for the number of conditions of facets that are integer. Woodward and Joe's method requires the investigator to specify an exact value for the generalizability coefficient which will almost always lead to non-integer solutions for the number of conditions of the facets. However, having an investigator specify a threshold for the generalizability coefficient, which is done in our method, is much more realistic. It is more likely that an investigator can specify a minimum value for a generalizability coefficient than a specific value for a generalizability coefficient.

The method proposed here employs ordinal, threshold and monotonicity constraints. Employing these constraints results in obtaining all the integer solutions which are in accord with the specified objective-function value and can be considered optimal solutions. On the other hand, Woodward and Joe's method implies a restrictive ratio constraint, a ratio of variance components, which results in precisely one optimal non-integer solution. For the two-facet example, rounding-off the optimal non-integer solution yields a solution equal to one of the optimal solutions, solution $(\hat{n}_1,\hat{n}_2)=(8,3)$, obtained by our method. Depending on the rounding-off, however, the optimal solutions of the two methods can differ as was demonstrated for the two-facet example in the introduction with $\rho^2 = .79$.

It has been shown that various constraints can be employed with the method proposed here. In general, the employment of equality constraints is discouraged because they will often lead to non-integer and/or non-optimal solutions. Woodward and Joe's method is an example where an equality constraint is used to specify an acceptable generalizability coefficient. With our method, employing in the two-facet example equality constraint $n_1=n_2$ as an economic constraint and specifying $g \geq .80$ will result in solution $(\hat{n}_1,\hat{n}_2)=(12,3)$ with $L = 36$ observations. However, $g \geq .80$ is also satisfied by solution $(\hat{n}_1,\hat{n}_2)=(8,3)$ with $L = 24$ observations. The latter solution is of course to be preferred because it needs four fewer conditons for facet 1. Employing more than one equality constraint will often even result in no integer solution at all, like when employing the two equality constraints $n_1=4n_2$ and $g = .80$ in the two-facet example. With respect to the number of constraints employed, it should be clear that adding constraints will reduce the set of feasible solutions.

The versatility of our method makes it extremely useful for investigators planning generalizability studies and practitioners involved with making decisions about the composition of measurement instruments. A computer program for the designs discussed here as well as other designs has been developed. Using as input G-study estimates of variance components obtainable from existing computer programs for generalizability studies, e.g., GENOVA (Crick and Brennan, 1982), computation time is no more than a few seconds.

The foregoing presentation has emphasized the practical aspects of the method proposed here. The method does however have important theoretical aspects as well. It can easily be seen that (2), the Spearman-Brown Prophecy formula from classical test theory, can be stated as a non-integer optimization problem with $L = n_1$ and $\rho^2(\hat{n}_1)=g$. By employing a threshold constraint and an integer constraint, this formula can be stated as an integer optimization problem. The search-tree of this problem consists of a root-node with one branch going down to one node at level 1, which is associated with the only feasible value that minimizes L. The method proposed in this article generalizes the Spearman-Brown formula for measurement instruments with one facet to measurement instruments with more than one facet. As generalizability theory is the theoretical framework for these instruments, this method is a theoretical contribution to this framework.

Thus far the number of applications of integer optimization techniques to solve problems in psychometrics has been limited. Theunissen (1985) was the first to show how integer optimization techniques could be used to solve problems in latent trait theory. For generalizability theory these techniques also appear ultimately suited to handle a broad range of practical problems. Decision studies no longer have to be conducted through successive approximations (Cardinet and Tourneur, 1985). As a result of optimization techniques the decision-study phase of generalizability theory will no longer be characterized by trial and error but will be a phase characterized by rational decision making. More fruitful applications of these techniques are to be expected in the near future (e.g., van der Linden and Boekkooi-Timminga, 1988).

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