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Abstract

The Partial Credit Model (PCM) is sometimes interpreted as a model for stepwise solution of polytomously scored items, where the item parameters are interpreted as difficulties of the steps. It is argued that this interpretation is not justified. A model for stepwise solution is discussed. It is shown that the PCM is suited to model sums of binary responses which are not supposed to be stochastically independent. As a practical result, a statistical test of stochastic independence in the Rasch model is derived.

Key words: Partial Credit Model, local stochastic independence, distribution of sums.

Introduction

Masters (1982) introduced the partial credit model (PCM), an IRT model for polytomous items with ordered categories. The rationale he used to introduce the model was based on a response process where the subject responds sequentially to a number of subproblems in the item. The partial credit given equals the number of steps completed successfully, which of course in this rationale should be the first steps. This rationale, together with the tempting conclusion that the location parameters in the PCM could be interpreted as difficulty parameters of the respective steps, was criticized by Molenaar (1983). The main point of this criticism can easily be illustrated from the definition of the PCM which states that for an item with maximum score m,

$$P(X=j \mid \theta, X=j \ or\ X=j-1) \propto \exp(\theta + \beta_j), \quad (j=1,\dots,m), \tag{1}$$

where θ is the latent variable and X is the item score. Suppose now that $j < m$, and consider the population of all persons with $\theta = \theta_0$. Suppose that step $j+1$ is infinitely difficult, and that the probability given by the left-hand side of (1) equals 0.5. Then half of the population with $\theta = \theta_0$ which has score at least $j-1$ will have score j, whence it follows that $\beta_j = -\theta_0$. But if step $j+1$ is very easy, then from the latter half a substantial proportion will have a score larger than j, while the proportion ending up with score $j-1$ will not change, since they do not try step $j-1$. It follows that the conditional probability expressed by the left-hand side of (1) will be smaller than .5, whence $\beta_j < \theta_0$. The conclusion should be that the parameter β_j does not depend on the difficulty of the j-th step alone but of the subsequent steps as well. Notice however that this criticism, and the conclusion is only valid if the sequential processing of the steps is accepted. But there is nothing in the formal derivation of the PCM which implies this sequential response process; only some introductory examples given by Masters suggested it, and the most one can say is that the PCM is not an adequate model for some of these example items.

Thus we are left with two problems. Since the sequential processing is very attractive, can we develop an IRT model which does model this process in an adequate way. The second problem concerns the PCM itself: if it is not suited for modeling sequential processes, then what is it good for: can it be put to use in a convincing way or should it be discarded despite its nice mathematical and statistical properties?

An answer to the first problem was found independently at two different places at about the same time. De Vries (1988) and Verhelst, Glas and De Vries (1997) developed a model by combining the simple Rasch model with a subject controlled incomplete design: the steps or subitems of a polytomously scored item are administered in a fixed sequence and the next subitem is presented if and only if the previous one was correctly responded to. The answer to each subitem is modeled by the simple Rasch model. The presentation of a subitem thus depends on the behavior of the responding subject, hence the qualification subject controlled. Tutz (1990, 1997) followed the same rationale, but introduced the model formally and more generally as

$$p_j \equiv P(X > j \mid \theta, X \geq j) = F(\theta + \beta_{j+1}), \quad (j = 0, \ldots, m-1), \tag{2}$$

where $F(.)$ is an arbitrary distribution function. It can readily be seen that in both models, the category response functions are given by

$$P(X = j \mid \theta) = \begin{cases} (1 - p_j) \prod_{g=0}^{j-1} p_g & \text{if } j < m, \\ \prod_{g=0}^{m-1} p_g & \text{if } j = m. \end{cases} \tag{3}$$

whence it follows that both models are identical if F is the logistic distribution function with argument $\theta + \beta_j$.

The remaining part of the paper concerns the demonstration that a generalized form of the PCM can always be interpreted as a unidimensional model of a sum of binary random variables, whose distributions follow a unidimensional IRT model.

The distribution of sums of Rasch item scores

Suppose $m(>1)$ items can be described by the Rasch model, i.e., for any value of the latent variable θ,

$$P(Y_i = y_i \mid \theta) \propto \exp\left[y_i(\theta + \beta_i)\right], \quad (i = 1, \ldots, m; y_i \in \{0, 1\}). \tag{4}$$

Defining the variable S as $S + \sum Y_i$, and assuming conditional independence as usual, it is readily seen that

$$P(S = s \mid \theta) \propto \exp(s\theta) \sum_{\Sigma y_i = s} \prod_i \varepsilon_i^{y_i}, \tag{5}$$

where $\varepsilon_i = \exp(\beta_i)$. The combinatorial function represented by the sum in the right-hand side of (5) is known as the basic symmetric function (of order s) of the multivariate argument $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_m)$, and will be denoted by $\gamma_s(\varepsilon)$. It is defined formally as

$$\gamma_s(\varepsilon) = \sum_{\Sigma y_i = s} \prod_i \varepsilon_i^{y_i} \quad \text{for } 0 \leq s \leq m. \tag{6}$$

Note that $\gamma_0(\varepsilon) = 1$. Defining

$$\eta_s = -\ln \gamma_s(\varepsilon), \quad (s = 0, \ldots, m), \tag{7}$$

equation (5) can be rewritten as

$$P(S = s \mid \theta) \propto \exp(s\theta - \eta_s), \tag{8}$$

which is nothing else than the category response function of the PCM in a parametrization first used by Andersen (1977). Notice that η_0 is zero.

If a test of k items follows the Rasch model, and this test is partitioned into T classes, consisting of $m_1, \ldots, m_T$ items, then the sums of the item scores in the classes can be described by the PCM because the original item responses are independent and the classes are disjoint. It will also be clear that, if the original items can be described by the two- parameter logistic model (2PLM), and if the discrimination parameters within the classes of the partition are constant, then the generalized PCM (Muraki, 1992, Verhelst & Glas, 1995) applies.

There are two important observations to be made in connection with this result. First, if only partial sums are observed instead of the original item scores, then it is possible to estimate the original Rasch parameters from the sum scores, but it is not possible to associate them with the original items. If all m β-parameters are distinct, then there are $m!$ different associations possible, and there is no way of distinguishing between them on the basis of the sum scores alone. The second observation is more important. Although it is true that sums of Rasch item scores are distributed following the PCM, the converse is not true: polytomous item scores whose distribution is given by the

PCM cannot always be interpreted as sums of Rasch item scores. In the PCM the parameter space (for one item) is $\mathbb{R}^m$, i.e., there are no restrictions on the parameters. To be interpretable as sums of Rasch item scores, however, a set of nonlinear restrictions must be fulfilled: there must exist positive ε-values such that (7) is fulfilled for the PCM parameters η. This means that it is not always possible to conceive of the PCM as a model of a sum of Rasch item scores. Van Engelenburg (1997) showed that for the case $m = 3$, certain inequalities must hold between the parameters of the PCM. His result can be generalized to arbitrary m, using the following theorem and corollary.

Theorem

The logarithm of the basic symmetric functions of k positive arguments $\varepsilon_1,\ldots,\varepsilon_k$, considered as a function of their order is strictly concave, i.e.,

$$\ln\gamma_s(\varepsilon) > \frac{\ln\gamma_{s-1}(\varepsilon) + \ln\gamma_{s+1}(\varepsilon)}{2}, \quad (s = 1,\ldots,k-1).$$

The proof is given in appendix. Since the logarithm is a strictly increasing function the corollary follows immediately.

Corollary 1

The basic symmetric functions of k positive arguments $\varepsilon_1,\ldots,\varepsilon_k$, considered as a function of their order, are single peaked in the following sense:

a) $\gamma_s \leq \gamma_{s+1} \Rightarrow \gamma_{s-1} < \gamma_s, \quad (s-1, s, s+1 \in D)$,

b) $\gamma_s \leq \gamma_{s-1} \Rightarrow \gamma_{s+1} < \gamma_s, \quad (s-1, s, s+1 \in D)$,

where $D = \{0, 1,\ldots,k\}$.

Corollary 1 shows that the restrictions (7) imply a number of inequality restrictions between the PCM parameters. These restrictions led Van Engelenburg to the conclusion that the PCM is not an adequate model to describe the distribution of sums of binary item scores. It will be shown in the next section that these restrictions are a direct consequence of assuming local independence between the binary item responses.

Models with dependent responses

To model dependencies between item responses, it is easier to model whole response patterns than merely item responses, because dependence means lack of local independence, and thus impossibility of multiplying item response functions. As before we assume that the test consists of k binary items, and is partitioned into T groups, containing $m_1, \ldots, m_t, \ldots, m_T$ items respectively. These groups will be called testlets. As most of the discussion to come will focus on a single testlet, explicit reference to the testlet number will be dropped.

Consider a testlet consisting of $m(>1)$ items. The vector $Y = (Y_1, \ldots, Y_m)$ with realization $y = (y_1, \ldots, y_m)$ will be called the response pattern. The random variable S, with realizations s, defined by

$$S \equiv S(Y) = \sum_i Y_i, \tag{9}$$

is called the testlet score. Define the m sets I_g, $g = 1, \ldots, m$, as the sets containing all g-tuples of item numbers. This means $I_1 = \{1, 2, \ldots, m\}$, $I_2 = \{(1,2), \ldots, (1,m), (2,3), \ldots, (m-1,m)\}$, etc. The cardinality of I_g is $\binom{m}{g}$. The general model that will be studied is given by

$$P(Y = y \mid \theta) \propto$$

$$\exp\left[s\theta + \sum_{I_1} y_i\beta_i + \sum_{I_2} y_i y_j \beta_{ij} + \sum_{I_3} y_i y_j y_\ell \beta_{ij\ell} + \cdots + \sum_{I_m} y_1 y_2 \cdots y_m \beta_{1,2,\ldots,m}\right]. \tag{10}$$

and by the assumption of independence between testlet response patterns. If all β parameters are free, the model is saturated. It is important to understand the kind of technical restrictions that must be imposed to make the model identifiable. Suppose the number of testlets in the test is 1. The number of β-parameters in the testlet is $\Sigma I_g = 2^m - 1$ and the number of different response patterns is 2^m, leaving $2^m - 1$ degrees of freedom. However, one degree of freedom must be used for fixing the origin of the scale, for example by choosing $\beta_1 = 0$. If there is more than one testlet, however, the origin of the scale can not be fixed more than once. So in the saturated model for all testlets taken jointly, the parameter space is $\mathbb{R}^M$, with

$$M = \sum_{t=1}^{T} (2^{m_t} - 1) - 1 = \sum_{t=1}^{T} 2^{m_t} - (T + 1). \tag{11}$$

Model (10) and several submodels resulting by setting interaction parameters to zero have been studied by Kelderman (1984); see also Verhelst and Glas, 1995. It should be stressed that model (10) and various submodels are estimable if the item responses are observed. What matters here, however, is to see what happens if only the testlet scores S_t are observed.

Taking sums of (10) for all response patterns with score s gives

$$P(S(Y) = s \mid \theta) \propto \exp(s\theta) \times$$

$$\sum_{\sum z_i = s} \exp\left[\sum_{I_1} z_i\beta_i + \sum_{I_2} z_i z_j \beta_{ij} + \sum_{I_3} z_i z_j z_\ell \beta_{ij\ell} + \cdots + \sum_{I_m} z_1 z_2 \cdots z_m \beta_{1,2,\ldots,m}\right]. \tag{12}$$

The second factor in the right-hand side of (12) is positive, and for given β-parameters its value only depends on s. Therefore this factor can be written as $\exp(-\eta_s)$. Moreover, it is clear from (12) that $\eta_0 = 0$. With this notation, (12) can be written as

$$P(S(Y) = s \mid \theta) \propto \exp(s\theta - \eta_s) \tag{13}$$

which is formally equivalent to the PCM. But the main question is whether for each PCM, i.e., every given vector of m η-parameters, a set of $2^m - 1$ β-parameters can be found, when inserted in (10) give (13) as a result with the fixed η-parameters. This question may seem trivial, because for $m > 1$, the number of β-parameters is larger than m, but we know from the preceding section that the parameter space of the PCM is not covered if in (10) all interaction parameters, i.e., all β-parameters with more than one subscript, are set to zero.

Since the second factor in the right-hand side of (12) defies simplification, a number of restrictions on the β-parameters will be introduced which yield a more tractable expression, and yet result in a model which covers the parameter space of the PCM. For instance, assume all interaction parameters of the same order to be equal:

$$\beta_h = \lambda_g \text{ for all } h \in I_g, \quad (g = 2, \ldots, m). \tag{14}$$

Using the restrictions (14) and the fact that all g-fold products $y_{i_1} y_{i_2} \dots y_{i_g}$ equal zero if $g > s(y)$, and equal one in $\binom{s}{g}$ cases is $g \leq s(y)$, (10) can be rewritten as

$$P(Y = y \mid \theta) \propto \exp\left[s\theta + \sum_{I_1} y_i \beta_i + \sum_{g=2}^{s} \binom{s}{g} \lambda_g\right], \tag{15}$$

whence it follows that

$$\begin{aligned} P(S(Y) = s \mid \theta) &\propto \exp(s\theta) \exp\left[\sum_{g=2}^{s} \binom{s}{g} \lambda_g\right] \sum_{\Sigma y_i = s} \prod_i \varepsilon_i^{y_i} \\ &= \exp(s\theta) \exp\left[\sum_{g=2}^{s} \binom{s}{g} \lambda_g\right] \gamma_s(\varepsilon) . \end{aligned} \tag{16}$$

Define

$$\eta_s = -\ln \gamma_s(\varepsilon) - \sum_{g=2}^{s} \binom{s}{g} \lambda_g, \quad (j = 1, \dots, m), \tag{17}$$

where the sum in the right-hand side of (17) is defined to be zero if $s < 2$. Now it is easy to show that for any ordered set of m η-values it is always possible to find $\varepsilon-$ and $\lambda-$values such that (17) is fulfilled. The values for the ε-parameters can be taken arbitrarily from the positive reals, with the only restriction that minus the logarithm of their sum equals η_1. In this way (17) is fulfilled for $s = 0$ and $s = 1$. The λ-values are given by sequentially applying (from (17)):

$$\lambda_s = -\ln \gamma_s - \eta_s - \sum_{g=2}^{s-1} \binom{s}{g} \lambda_g, \quad (s = 2, \dots, m). \tag{18}$$

In summary, it has been shown that every model in the family defined by (10) is formally equivalent to the PCM when the distribution of the testlet score is modeled, and conversely, every PCM can be understood as a model for the testlet score, where the joint distribution of the item responses within the testlet is given by (10).

If the item responses are observed, then (15) is identified and the parameters may be estimated; if only sums of item scores are observed, however, model (16) results, and the model is no longer identifiable, because there are more parameters than different values of the score. Only functions of these parameters are estimable, viz. the functions given by (17) and one-one transformations of these functions. This may sound a bit disappointing, but it should be understood that the restrictions (14) were introduced only to be able to prove that the parameter space of the PCM is covered by the parameter space of model (10).

An interesting, but different question is whether it is possible to find a submodel of (12) which covers the parameter space of the PCM and has exactly m free parameters. It might be possible, for example, to create such a model by fixing all β-parameters except for m well chosen main effect or interaction parameters. The answer to this question is positive and the proof trivially follows from the preceding result. Any set of $m - 1$ linearly independent linear restrictions on the main effect parameters (β_i) which are also independent from the restriction $\Sigma\beta_i = \exp(-\eta_1)$, corresponds to a particular choice of the ε-parameters such that (17) can be solved for any value of η_1, and the values of all λ-parameters can be solved from (18) for any value of the remaining η-parameters. As an example, the submodel of (10) which results from applying (14) and the $m - 1$ additional restrictions $\beta_i = \beta_1, i = 2, \dots, m$, is identified, and every PCM can be interpreted as being a member of this restricted family.

Interestingly, if additionally to the restrictions (14), the restriction $\lambda_h = c$, for some h in $\{2, \dots, m\}$ is added, the resulting model does not any longer cover the parameter space of the PCM. The proof is very simple.

Assume $\lambda_h = c$ for some h in $\{2, \dots, m\}$ and choose $\eta_1 \geq 0$. Then it follows from (17) that $\gamma_1 \leq 1 = \gamma_0$, whence, from the corollary, $\gamma_h < \gamma_{h-1}$. Applying (17) and Corollary 1 gives

$$
\begin{aligned}
\eta_h &= -\ln\gamma_h - \sum_{g=2}^{h-1}\binom{h}{g} - \lambda_h \\
&> -\ln\gamma_{h-1} - \sum_{g=2}^{h-1}\binom{h}{g}\lambda_g - \lambda_h \\
&= \eta_{h-1} - \sum_{g=2}^{h-1}\binom{h-1}{g-1}\lambda_g - c,
\end{aligned}
$$

meaning that the difference $\eta_h - \eta_{h-1}$ is bounded.

This result implies an interesting relationship between the PCM and the Rasch model. Setting all λ-parameters to zero, together with restriction (14) implies stochastic independence between the m item responses in the testlet. To get rid of the condition that $\eta_1 \geq 0$ in the preceding result, one can apply the Theorem and (17) directly to obtain

Corollary 2

A necessary condition for the PCM to be interpretable as the distribution of the sum of m Rasch item scores is that the $m + 1$ parameters $\eta_0, \ldots, \eta_m$ are a strict convex function of their index.

In applications of the Rasch model, Corollary 2 offers a nice opportunity to test the assumption of local independence. For an arbitrary (proper) subset of the binary items, the sum score can be substituted for the original binary responses. Suppose there is a subset containing m items. With suitable estimators of the parameters (e.g., CML estimators of the PCM) the $m - 1$ null hypotheses

$$\eta_s < \frac{\eta_{s-1} + \eta_{s+1}}{2}, \quad (s = 1, \ldots, m-1) \tag{19}$$

can be tested by a series of one degree of freedom Wald tests. The power of this test will probably not be very high. Inequality (19) immediately derives from the Theorem, but sharper inequalities are possible. For example, if $m = 2$, it is easy to verify that

$$\gamma_1^2 = (\varepsilon_1 + \varepsilon_2)^2 \geq 4\,\varepsilon_1\,\varepsilon_2 = 4\,\gamma_0\,\gamma_2,$$

which corresponds, using (7), to the null hypothesis

$$\eta_1 \leq \frac{\eta_2 - \ln 4}{2}.$$

Discussion

The PCM with testlet scores ranging from 0 to m_t, can always be written as a sum of m binary item scores, where the item response distribution is given by (15). It is shown that if only such sums are observed, the original model is not identified. For every possible value in the parameter space of the PCM arbitrary β-parameters can be chosen, as long as restriction (17) for $s = 1$ is fulfilled. An example is given of a class of identified models which covers the whole parameter space of the PCM. But notice that there might be other models with the same characteristics and with a different interpretation. Even more dramatically, the PCM can offer an accurate description of the weighted sum of p item scores where $p < m$. Suppose that for $p = 3$ items the two parameter logistic model is valid, i.e.,

$$P(Y_i = y_i \mid \theta) \propto \exp\left[\alpha_i y_i(\theta + \beta_i)\right], \quad (i = 1,\ldots,3\,;\, y_i \in \{0, 1\}),$$

and that the weights (the discrimination parameters) α_i for the items are 1, 1 and 2 respectively. Moreover local independence is assumed. The distribution of the weighted sum score, W, is given by

$$P(W = w \mid \theta) \propto \exp\left[w\theta + \ln\tilde{\gamma}_w(\varepsilon)\right], \quad (w = 1, \ldots, 4),$$

where $\varepsilon_i = \exp(\alpha_i \beta_i)$, and

$$\tilde{\gamma}_0 = 1$$

$$\tilde{\gamma}_1 = \varepsilon_1 + \varepsilon_2,$$

$$\tilde{\gamma}_2 = \varepsilon_1\varepsilon_2 + \varepsilon_3,$$

$$\tilde{\gamma}_3 = \varepsilon_3(\varepsilon_1 + \varepsilon_2),$$

$$\tilde{\gamma}_4 = \varepsilon_1\varepsilon_2\varepsilon_3.$$

This means that a well fitting PCM can have many interpretations, and none of them can be preferred on the basis of the observed polytomous scores. But the practical importance of the results presented here stems from the converse situation: if there is evidence (or mere suspicion) that for a subset of binary items the assumption of local independence is violated, taking sums of item scores over this subset neutralizes the effect of any interaction parameters in a class of models as broad as the one given by (12). If the main purpose of the model construction is to determine θ as accurate as possible, no information with respect to θ is lost if local independence is not violated; if it is violated, the embarrassing implications are avoided by considering sums of item scores.

Appendix

Proof of the Theorem

Theorem

The logarithm of the basic symmetric functions of k positive arguments $\varepsilon_1, \ldots, \varepsilon_k$, considered as a function of their order is strictly concave, i.e.,

$$\ln\gamma_s(\varepsilon) > \frac{\ln\gamma_{s-1}(\varepsilon) + \ln\gamma_{s+1}(\varepsilon)}{2}, \quad (s = 1, \ldots, k-1).$$

Proof

In the proof, the argument ε will be dropped. The inequality stated in the Theorem is equivalent to

$$\gamma_s^2 > \gamma_{s-1}\gamma_{s+1}, \quad (s = 1, \ldots, k-1). \tag{A1}$$

Both sides of the inequality (A1) consist of a sum of products, each term having $2s$ factors. Each term can be written generically as

$$t = \varepsilon_{i_1}^2 \varepsilon_{i_2}^2 \ldots \varepsilon_{i_j}^2 \varepsilon_{i_{j+1}} \ldots \varepsilon_{i_s} \varepsilon_{i_{s+1}} \ldots \varepsilon_{i_{2s-j}}, \tag{A2}$$

where all indices i_g, $(g = 1, \ldots, 2s - j)$ are different from each other. So each term t can be characterized by two sets of indices, J_t and I_t, the set J_t containing the indices in (A2) which appear in factors which are squared and I_t containing the remaining indices. The ordered pair (I_t, J_t) will be called the signature of term t. Obviously, terms with the same signature are identical. Each pair (I, J) of sets such that

$$\begin{cases} I, J \subseteq \{1, \ldots, k\}, \\ I \cap J = \varnothing, \\ \#I + 2(\#J) = 2s \end{cases} \tag{A3}$$

is the signature of a number of terms in the expansion of both sides of (A1). Let $R(I, J)$ denote the multiplicity of the term with signature (I, J) in the right-hand side of inequality (A1), and $L(I, J)$ the multiplicity in the left-hand side, and let $j = \#J.L(I, J)$ is given by the number of ways the $2(s-j)$ indices in I can be partitioned into to subsets of $s-j$ elements, if $j < s$. If $j = s$, the set I is empty. Clearly

$$L(I,J) = \begin{cases} \begin{bmatrix} 2(s-j) \\ s-j \end{bmatrix} & \text{if } j < s, \\ 1 & \text{if } j = s. \end{cases}$$

For the right-hand side, the $2(s-j)$ elements of I have to be partitioned into two subsets containing $s-j-1$ and $s-j+1$ elements if $j < s$. If $j = s$, $R(I,J)$ equals zero because in the expansion of γ_{s-1} and γ_{s+1}, the terms have $s-1$ and $s+1$ factors respectively. Therefore

$$R(I,J) = \begin{cases} \begin{bmatrix} 2(s-j) \\ s-j-1 \end{bmatrix} & \text{if } j < s, \\ 0 & \text{if } j = s. \end{cases}$$

and it follows that for all pairs (I,J), $(L(I,J) > R(I,J)$, and since all terms are positive, (A1) is true. □

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